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Three-term asymptotic expansions for the moments of the random walk with triangular distributed interference of chance

Rovshan Aliyev [a,c], Zafer Kucuk [a,*], Tahir Khaniyev [b,d]

[a] Karadeniz Technical University, Faculty of Arts and Sciences, Department of Statistics and Computer Sciences, 61080 Trabzon, Turkey
[b] TOBB University of Economics and Technology, Faculty of Engineering, Department of Industrial Engineering, 06560 Sogutozu, Ankara, Turkey
[c] Baku State University, Faculty of Applied Mathematics and Cybernetics, Department of Probability Theory and Mathematical Statistics, Baku, Azerbaijan
[d] Institute of Cybernetics of Azerbaijan National Academy of Sciences, Baku, Azerbaijan

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ABSTRACT

In this study, a semi-Markovian random walk with a discrete interference of chance ($X(t)$) is considered and under some weak assumptions the ergodicity of this process is discussed. The exact formulas for the first four moments of ergodic distribution of the process $X(t)$ are obtained when the random variable ζ_1, which is describing a discrete interference of chance, has a triangular distribution in the interval $[s, S]$ with center $(S+s)/2$. Based on these results, the asymptotic expansions with three-term are obtained for the first four moments of the ergodic distribution of $X(t)$, as $a \equiv (S-s)/2 \to \infty$. Furthermore, the asymptotic expansions for the variance, skewness and kurtosis of the ergodic distribution of the process $X(t)$ are established. Finally, by using Monte Carlo experiments it is shown that the given approximating formulas provide high accuracy even for small values of parameter a.

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1. Introduction

Many interesting problems resulting from the theories of stock control, reliability, queuing, mathematical biology, stochastic finance, mathematical insurance and so on can be expressed by random walk or by the help of the modification of this process. There are many valuable studies on these processes in the literature (for examples [1–9]). However, the existing studies are generally theoretical and they are not exactly helpful to solve concrete problems in practice due to the complexity of their mathematical formulas. Therefore, in addition to exact formulas, several approximated formulas are offered for these kinds of problems in the literature (for examples [1,2,4,8–13]). The approximated formulas are generally simpler and easier in application. But, on the other hand, it is necessary that the approximated formulas should be sufficiently close to the exact expressions. One of the most effective methods to obtain these kinds of approximated formulas is the asymptotic method. In many cases it is possible to obtain approximated formulas, which are closer to the exact expressions, by increasing the number of terms in the obtained asymptotic expansions. However, when the number of terms in the asymptotic expansions is considerably increased, the approximated expressions start to lose their simplicity and meaning. Therefore, in this study we consider only three-term asymptotic expansions. Let us give the following models before going to analyze the main problem.

* Corresponding author
E-mail addresses: zkucuk1@yahoo.com (Z. Kucuk), tahirkhaniyev@etu.edu.tr (T. Khaniyev).

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1.1. The model

Suppose that, any system is in state $z = s + x$ at initial time $t = 0$. Here, $s > 0$ is a predefined control level and $x \geqslant 0$. Signals of demands and supplies are included in the system at random times $T_n = \sum_{i=1}^{n} \xi_i, n \geq 1$. System passes from one state ($X(t)$) to another by jumping at time T_n, according to quantities of demands or supplies $\{-\eta_n\}$, $n \geqslant 1$ as follows:

$$X(T_1) \equiv X_1 = z - \eta_1, \quad X(T_2) \equiv X_2 = z - \eta_1 - \eta_2, \ldots, X(T_n) \equiv X_n = z - \eta_1 - \eta_2 - \ldots - \eta_n, \ldots$$

This variation of system continues until a certain random time τ_1, which is referred to as the first crossing time from the control level $s > 0$. When the system crosses the control level $s > 0$, the system is immediately brought to the position ζ_1, where the random variable ζ_1 has a certain distribution in the interval $[s, S]$. We define one period as the time between two consecutive epochs at which the state of system ($X(t)$) crosses $s > 0$. According to this definition, first period ends at τ_1, the second ends at τ_2, and so on. Then,

$$X(\tau_n + 0) = \zeta_n, \quad n = 1, 2, 3, \ldots$$

Here ζ_n 's had the same distribution as ζ_1.

Our aim, in this paper, is to construct a stochastic process $X(t)$, which mathematically describes the model above and to obtain three-term asymptotic expansions for first the four ergodic moments of the process $X(t)$. Moreover, for testing the accuracy of the obtained approximation formulas, we will use the Monte Carlo simulation exact expressions. Note that analogically stochastic process with exponentionally distributed interference of chance is investigated in the study [10]. But in the present study the random variable ζ_1 which describes a discrete interference of chance has a triangular distribution in the interval $[s, S]$ with center $(S + s)/2$. In other words, it is desired that ζ_1 takes values close to S and s with a very low probability. Having ζ_1 very close to S increases the average inventory level, which, in turn, increases the holding cost. On the other hand, if ζ_1 takes values close to s, then the system may re-start again in very short time intervals. This is not desirable in practice as it increases the number of orders, which causes an increase in the total ordering cost.

The following specific (real) model explain why the random variable ζ_1 which describes a discrete interference of chance may be a triangular distribution in the interval $[s, S]$.

1.2. Real model

A company operating in the energy sector produces, stores, fills and distributes liquefied petroleum gas (LPG). Domestic LPG distribution is carried out through pipelines and land transport. Where there is no pipeline installation, gas is distributed through land transport. LPG is carried from the LPG production center (a city in Turkey) to the 30 dealers by tankers with the capacities of 22 m^3 (approx. 10–11 tons) and 35 m^3 (approx. 17–18 tons). The tankers are kept under surveillance with Global Positioning System (GPS) 24 h and seven days. After delivering the needed amount of gas to the dealer, if more than 10% of the capacity of the tanker is left over, the tanker waits in its position until the next order of the dealer. Each dealer has a storage capacity of $S = 30$ m^3 (one storage tank of 30 m^3 or two storage tanks 10 and 20 m^3). Random amounts of LPG (η_n) are sold from these storage tanks at random times (ξ_n). When at random moments τ_n, $n \geqslant 1$ the level of LPG in the tank of the dealer falls below the control level $s = S/5$, a demand signal is automatically sent online to the production center. As a response to this demand, the nearest tanker(s) to the dealer is directed to the demanding dealer. If there is no tanker near the dealer, a full tanker is sent from the production center.

For security concerns (in order not to let the gas pressure reach its maximum value), the dealers usually fill 85% of the capacity (S) of their tanks. However, by taking a risk the dealers fill their tanks to the full capacity when need arises. On the other hand, even if the amount of gas in the tanker does not meet 85% of the dealer's tank, the amount of gas in the tanker is loaded into the dealer's tank.

To sum up the working principle explained above, after each filling, 85% of the tank of each dealer is most probably filled. With a remote possibility, the capacities of the tanks are used to their lowest limit s or to the highest limit S.

Therefore, in our opinion, the process which explains the working of the storage explained above can be considered as a stochastic process with a triangular distributed interference of chance.

Now, let us construct the process $X(t)$ from the mathematical point of view.

2. Mathematical construction of the process $X(t)$

Let $\{(\xi_n, \eta_n, \zeta_n)\}$, $n = 1, 2, 3,...$, be a sequence of independent and identically distributed triples of random variables defined on any probability space $(\Omega, \Im, P)$, such that ξ_n's are positive, η_n's take negative values as well as positive ones, ζ_n 's have a triangular distribution in the interval $[s, S]$, with center $(S + s)/2$. Suppose that ξ_1, η_1, ζ_1 are mutually independent random variables and their distribution functions are known, i.e.

$$\Phi(t) = P\{\xi_1 \leq t\}, \quad F(x) = P\{\eta_1 \leq x\}, \quad \pi(z) = P\{\zeta_1 \leq z\}, \quad t > 0,\ x > 0,\ z \in [s, S].$$

Define renewal sequence $\{T_n\}$ and random walk $\{S_n\}$ as follows:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad S_n = \sum_{i=1}^{n} \eta_i, \quad T_0 = S_0 = 0, \ n = 1, 2, \ldots$$

Introduce a sequence of integer valued random variables $\{N_n\}$ as:

$$N_0 = 0; \quad N_{n+1} = \inf\{k \geq N_n + 1 : \zeta_n - S_k + S_{N_n} < s\}, \quad n \geq 0, \quad \zeta_0 = z \in [s, S].$$

Here $\inf\{\varnothing\} = +\infty$ is stipulated.

Put $\tau_n = T_{N_n}$, $n \geqslant 0$ and define the $\nu(t)$ as:

$$\nu(t) = \max\{n \geq 0 : T_n \leq t\}.$$

We can now construct desired stochastic process $X(t)$ as follows:

$$X(t) = \zeta_n - S_{\nu(t)} + S_{N_n}, \quad \text{if} \quad \tau_n \leq t < \tau_{n+1}, \quad n = 0, 1, 2, \ldots; \quad \zeta_0 = z \in [s, S].$$

The process $X(t)$ is called a semi-Markovian random walk with triangular distributed interference of chance. One of the trajectories for the process $X(t)$ as shown in Fig. 1:

The main aim of this study is to investigate the asymptotic behaviour of the stationary moments of $X(t)$ as $a \equiv (S - s)/2 \to \infty$. For this purpose, we first discuss the ergodicity of the process $X(t)$.

Introduce the following notations:

$$a_n(x, z) = P\{z - S_k \geq s, k = \overline{1, n}; z - S_n \leq x\}, \ n \geq 1;$$
$$a_0(x, z) = \varepsilon(x - z);$$

where $\varepsilon(t) = 1$ as $t \geqslant 0$ and $\varepsilon(t) = 0$ as $t < 0$;

$$A(x, z) = \sum_{n=0}^{\infty} a_n(x, z); \quad A(x, \cdot) = \int_s^S A(x, z) d\pi(z).$$

3. Preliminary discussions

To investigate the stationary characteristics of the considered process, it is necessary to prove that $X(t)$ is ergodic under some assumptions. This property can be given by the following proposition.

Proposition 3.1. *Let the initial sequence of the random variables* $\{(\xi_n, \eta_n, \zeta_n)\}, n \geqslant 1$ *satisfy the following supplementary conditions*:

(i) $0 < E\xi_1 < \infty$, (ii) $0 < E\eta_1 < \infty$, (iii) η_1 *is non-arithmetic random variables.*

Then the process $X(t)$ *is ergodic and the following expression is correct with probability* 1, *for each measurable bounded function* $f(x)$ $(f{:}[s, S] \to R)$:

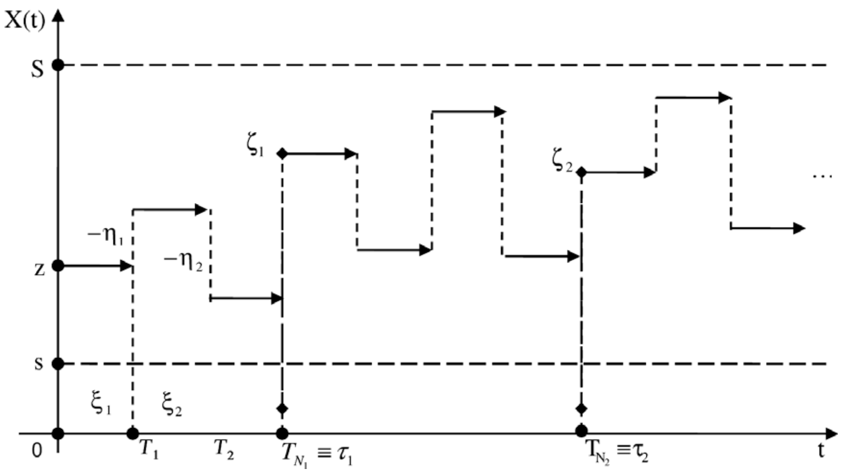

Fig. 1. One of the trajectories of the process $X(t)$.

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(u))du = \frac{1}{A(\infty,\cdot)}\int_s^S f(x)d_xA(x,\cdot), \tag{3.1}$$

where $A(\infty,\cdot)\equiv\lim_{x\to\infty}A(x,\cdot)$.

Proof. The process $X(t)$ belongs to wide class of the processes which is known in the literature as a class of the semi-Markov processes with a discrete interference of chance. Furthermore, the ergodic theorem of type Smith's "key renewal theorem" exists in the literature for this class (see, Gihman and Skorohod [14], p. 243). Since the assumptions of this general theorem have been provided under the conditions of Proposition 3.1, the process $X(t)$ is ergodic and Eq. (3.1) have been provided with a probability 1. □

Let us put $\varphi_X(\alpha)\equiv\lim_{t\to\infty}E\{\exp(i\alpha X(t))\}, \alpha\in R$.

Using the basic identity for random walks (see, Feller [15], p. 514) and Proposition 3.1, we obtain the following Proposition 3.2.

Proposition 3.2. *Suppose that the assumptions of* Proposition 3.1 *are satisfied and random variable* ζ_1, *which is describes a discrete interference of chance, has a triangular distribution in the interval* $[s, S]$ *with center* $(S+s)/2$. *Then the characteristic function* $\varphi_x(\alpha)$ *of the ergodic distribution of the process* $X(t)$ *can be expressed by means of the characteristics of the pair* $(N(x), S_{N(x)})$ *and random variable* η_1 *as follows*:

$$\varphi_X(\alpha)\equiv\lim_{t\to\infty}E\{e^{i\alpha X(t)}\} = \frac{1}{EN}\int_s^S e^{i\alpha z}\frac{\varphi_{S_{N(z-s)}}(-\alpha)-1}{\varphi_\eta(-\alpha)-1}d\pi(z), \tag{3.2}$$

where

$$EN = \int_s^S EN_1(z-s)d\pi(z); \quad \varphi_{S_N}(\alpha) = E\exp(i\alpha S_{N_1});$$
$$\varphi_\eta(\alpha) = E\exp(i\alpha\eta_1), \quad \alpha\in R\setminus\{0\}.$$

4. Exact formulas for the first four moments of the ergodic distribution

The aim of this section is to express the first four moments of the ergodic distribution of the process $X(t)$ by the appropriating ones of the boundary functional $S_{N(z-s)}$ and random variable η_1. For this, let us introduce the following notations:

$$m_k = E(\eta_1^k); \quad M_k(x) = E\left(S^k_{N(x)}\right), \quad k=\overline{1,5}, \quad x>0$$

and for the shortness of expressions we put

$$m_{k1} = \frac{m_k}{m_1}, \quad M_{k1}(x) = \frac{M_k(x)}{M_1(x)}, \quad k=\overline{2,5};$$
$$J_{kn}(a) = \int_0^{2a} x^n M_k(x)p_a(x)dx, \quad k=\overline{1,5}; \quad n=\overline{0,4}; \quad a\equiv(S-s)/2;$$
$$p_a(x) = \pi'(x+s) = \frac{x}{a^2}, \quad \text{if } 0\le x<a \quad \text{and} \quad p_a(x)=\pi'(x+s) = \frac{2a-x}{a^2} \quad \text{if } a\le x\le 2a,$$
$$E(\overline{X}^k) = \lim_{t\to\infty}E((\overline{X}(t))^k), \quad k=\overline{1,4},$$

where $\overline{X}(t)\equiv X(t)-s$.

For the measurable and bounded function $M(x)$:
$c^*M(x) = c\int_0^x M(u)du,\ x>0$, where, c is constant.
We can now state the first main result of this section as follows:

Theorem 4.1. *Suppose that the conditions of the* Proposition 3.2 *and* $E|\eta_1|^3<\infty$ *are satisfied. Then the first and second moments of the ergodic distribution of the process* $\overline{X}(t)$ *exist and can be expressed by means of the characteristics of the boundary functional* $S_{N(x)}$ *and random variable* η_1 *as follows*:

$$E(\overline{X}) = \frac{2J_{11}(a)-J_{20}(a)}{2J_{10}(a)} + A_1; \tag{4.1}$$
$$E(\overline{X}^2) = \frac{1}{J_{10}(a)}\left\{J_{12}(a)-J_{21}(a)+m_{21}\left[J_{11}(a)-\frac{1}{2}J_{20}(a)\right]+\frac{1}{3}J_{30}(a)\right\}+A_2, \tag{4.2}$$

where

$$A_1 = \frac{m_{21}}{2}, \quad A_2 = \frac{m_{21}^2}{2}-\frac{m_{31}}{3}.$$

Theorem 4.2. *Suppose that the conditions of the Proposition 3.2 and $E|\eta_1|^5<\infty$ are satisfied. Then the third and fourth moments of the ergodic distribution of the process $X(t)$ exist and can be expressed by means of the characteristics of the boundary functional $S_{N(x)}$ and random variable η_1 as follows*:

$$E(\overline{X}^3)=\frac{1}{J_{10}(a)}\left\{J_{13}(a)-\frac{3}{2}J_{22}(a)+J_{31}(a)-\frac{1}{4}J_{40}(a)+3A_1\left[J_{12}(a)-J_{21}(a)+\frac{1}{3}J_{30}(a)\right]+3A_2\left[J_{11}(a)-\frac{1}{2}J_{20}(a)\right]\right\}+3A_3;$$

$$E(\overline{X}^4)=\frac{1}{J_{10}(a)}\left\{J_{14}(a)-2J_{23}(a)+2J_{32}(a)-J_{41}(a)+\frac{1}{5}J_{50}(a)+4A_1\left[J_{13}(a)-\frac{3}{2}J_{22}(a)+J_{31}(a)-\frac{1}{4}J_{40}(a)\right]\right.$$
$$\left.+6A_2\left[J_{12}(a)-J_{21}(a)+\frac{1}{3}J_{30}(a)\right]+12A_3\left[J_{11}(a)-\frac{1}{2}J_{20}(a)\right]\right\}+3A_4,$$

where

$$A_3=\frac{m_{41}}{12}-\frac{m_{31}m_{21}}{3}+\frac{m_{21}^3}{4},$$
$$A_4=\frac{m_{21}^4}{4}-\frac{m_{31}m_{21}^2}{2}+\frac{m_{41}m_{21}}{6}+\frac{m_{31}^2}{9}-\frac{m_{51}}{30}.$$

4.1. Proof of Theorems 4.1 and 4.2

Note that the conditions of Theorems 4.1 and 4.2 provide the existence and finiteness of first five moments of $S_{N(x)}$ (see, Feller [15], p. 514). Therefore, Taylor expansions of the characteristic functions $\varphi_\eta(-\alpha)$ and $\varphi_{S_{N(x)}}(-\alpha)$ can be obtained, as $\alpha\to 0$. Using that this Taylor expansions in (3.2) it can be obtained the statements of the Theorems 4.1 and 4.2.

5. Third-order asymptotic expansions for the first four moments of the ergodic distribution

In this section, we obtain three-term asymptotic expansions for the first four moments of the ergodic distribution of the process $X(t)$. For this aim, we use ladder variables of a random walk. Consider the random walk $S_n=\sum_{i=1}^n\eta_i, n\geq 1$, with initial state $S_0=0$. Let $\nu_1^+=\min\{n\geq 1: S_n>0\},\ \chi_1^+=S_{\nu_1^+}$.

Note that, the random variables ν_1^+ and χ_1^+ is called the first strict ascending ladder epoch and ladder height of the random walk $\{S_n\}, n\geqslant 0$, respectively (see, Feller [15], p. 391).

Let $\{\chi_n^+\}, n\geq 1$ be a sequence of independent and identically distributed variables, having the same common distribution with the random variable χ_1^+.

Define $H(x)=\min\{n\geq 1:\sum_{i=1}^n\chi_i^+\geq x\},\ x\geq 0$.

Note that $H(x)$ is a renewal process, which is generated by means of the positive valued random variables $\chi_n^+, n\geq 1$. According to Dynkin's principle, it can be shown that

$$N(x)=\sum_{i=1}^{H(x)}\nu_i^+\quad\text{and } S_{N(x)}=\sum_{i=1}^{H(x)}\chi_i^+.$$

The following Lemma 5.1 is given in the paper [9]:

Lemma 5.1 (Khaniyev and Mammadova [9]). *Let the conditions of Theorem 4.1 be satisfied. Then the following asymptotic expansions are true for the first five moments of $S_{N(x)}$, as $x\to\infty$*:

(1) $M_1(x)=x+\frac{\mu_{21}}{2}+o(\frac{1}{x})$;	(2) $M_2(x)=x^2+\mu_{21}x+\frac{1}{3}\mu_{31}+o(1)$;
(3) $M_3(x)=x^3+\frac{3}{2}\mu_{21}x^2+\mu_{31}x+o(x)$;	(4) $M_4(x)=x^4+2\mu_{21}x^3+2\mu_{31}x^2+o(x^2)$;
(5) $M_5(x)=x^5+\frac{5}{2}\mu_{21}x^4+\frac{10}{3}\mu_{31}x^3+o(x^3)$;	

where $\mu_k=E(\chi_1^+)^k, \mu_{k1}=\mu_k/\mu_1, k=2,3; M_k(x)=E\left(S_{N(x)}^k\right), k=\overline{1,5}$.

Corollary 5.1. *Under the assumptions of Lemma 5.1, the following asymptotic expansions are true for the integrals from the moments of $S_{N(x)}$, as $x\to\infty$*:

(1) $1^*M_1(x)=\frac{1}{2}x^2+\frac{1}{2}\mu_{21}x+\frac{1}{12}[3\mu_{21}^2-2\mu_{31}]+o(1)$;
(2) $1^*(x^kM_1(x))=\frac{x^{k+2}}{k+2}+\frac{x^{k+1}}{2(k+1)}\mu_{21}+o(x^k), k\geq 1$;
(3) $1^*(x^kM_2(x))=\frac{x^{k+3}}{k+3}+\frac{x^{k+2}}{k+2}\mu_{21}+\frac{x^{k+1}}{3(k+1)}\mu_{31}+o(x^{k+1}), k\geq 0$;

(4) $1^*(x^kM_3(x)) = \frac{x^{k+4}}{k+4} + \frac{3x^{k+3}}{2(k+3)}\mu_{21} + \frac{x^{k+2}}{k+2}\mu_{31} + o(x^{k+2}),\ k \geq 0;$
(5) $1^*M_4(x) = \frac{1}{5}x^5 + \frac{1}{2}\mu_{21}x^4 + \frac{2}{3}\mu_{31}x^3 + o(x^3);$
(6) $1^*(xM_4(x)) = \frac{1}{6}x^6 + \frac{2}{5}\mu_{21}x^5 + \frac{1}{2}\mu_{31}x^4 + o(x^4);$
(7) $1^*M_5(x) = \frac{1}{6}x^6 + \frac{1}{2}\mu_{21}x^5 + \frac{5}{6}\mu_{31}x^4 + o(x^4);$
(8) $1^*(xM_5(x)) = \frac{1}{7}x^7 + \frac{5}{12}\mu_{21}x^6 + \frac{2}{3}\mu_{31}x^5 + o(x^5).$

Lemma 5.2. *Under the assumptions of Theorem 4.1 the following asymptotic expansions are true as $a \to \infty$:*

$J_{10}(a) = a + \frac{1}{2}\mu_{21} + o(1/a);$	$J_{11}(a) = \frac{7}{6}a^2 + \frac{1}{2}\mu_{21}a + o(1);$
$J_{12}(a) = \frac{3}{2}a^3 + \frac{1}{2}\mu_{21}a^2 + o(a);$	$J_{13}(a) = \frac{31}{15}a^4 + \frac{3}{4}\mu_{21}a^3 + o(a^2);$
$J_{14}(a) = 3a^5 + \frac{31}{30}\mu_{21}a^4 + o(a^3);$	$J_{20}(a) = \frac{7}{6}a^2 + \mu_{21}a + \frac{5}{3}\mu_{31} + o(1);$
$J_{21}(a) = \frac{3}{2}a^3 + \frac{7}{6}\mu_{21}a^2 + o(a);$	$J_{22}(a) = \frac{31}{15}a^4 + \frac{3}{2}\mu_{21}a^3 + \frac{7}{18}\mu_{31}a^2 + o(a^2);$
$J_{23}(a) = 3a^5 + \frac{31}{15}\mu_{21}a^4 + \frac{1}{2}\mu_{31}a^3 + o(a^3);$	$J_{30}(a) = \frac{5}{2}a^3 + \frac{15}{4}\mu_{21}a^2 + o(a);$
$J_{31}(a) = \frac{31}{15}a^4 + \frac{9}{4}\mu_{21}a^3 + o(a^2);$	$J_{32}(a) = 3a^5 + \frac{31}{10}\mu_{21}a^4 + \frac{3}{2}\mu_{31}a^3 + o(a^3);$
$J_{40}(a) = \frac{31}{15}a^4 + 3\mu_{21}a^3 + \frac{7}{3}\mu_{31}a^2 + o(a^2);$	$J_{41}(a) = 3a^5 + \frac{62}{15}\mu_{21}a^4 + 3\mu_{31}a^3 + o(a^3);$
$J_{50}(a) = 3a^5 + \frac{31}{6}\mu_{21}a^4 + 5\mu_{31}a^3 + o(a^3).$	

Proof. Proof of *Lemma* 5.2 implies from the *Lemma* 5.1 and *Corollary* 5.1 using by integration methods from studies of Federyuk (see, Federyuk [16]). □

Theorem 5.1. *Let the conditions of Theorem 4.1 be satisfied. Then the following asymptotic expansion can be written for the first two moments of the ergodic distribution of the process $X(t)$, as $a \to \infty$:*

$$E(\overline{X}) = \frac{7}{12}a + \frac{1}{2}\left(m_{21} - \frac{7}{12}\mu_{21}\right) + \frac{1}{6}\left(\frac{7}{8}\mu_{21}^2 - 5\mu_{31}\right)\frac{1}{a} + o\left(\frac{1}{a}\right); \tag{5.1}$$

$$E(\overline{X}^2) = \frac{1}{2}a^2 + \frac{1}{12}(7m_{21} - 3\mu_{21})a + \frac{m_{21}^2}{2} - \frac{m_{31}}{3} - \frac{7}{24}m_{21}\mu_{21} + \frac{1}{8}\mu_{21}^2 + o(1), \tag{5.2}$$

where $E(\overline{X}^k) \equiv \lim_{t\to\infty} E(X(t) - s)^k,\ k \geq 1.$

Proof. According to *Lemma* 5.2, as $a \to \infty$ □

$$\frac{1}{2J_{10}(a)} = \frac{1}{2a}\left[1 - \frac{1}{2}\mu_{21}\frac{1}{a} + \frac{1}{4}\mu_{21}^2\frac{1}{a^2} - \frac{1}{8}\mu_{21}^3\frac{1}{a^3} + o\left(\frac{1}{a^3}\right)\right]. \tag{5.3}$$

The other hand, as $a \to \infty$

$$2J_{11}(a) - J_{20}(a) = \frac{7}{6}a^2 - \frac{5}{3}\mu_{31} + o(1). \tag{5.4}$$

Substituting the expressions (5.3) and (5.4) in to formula (4.1) we get the asymptotic expansion (5.1). Analogically substituting the asymptotic expansions $J_{12}(a), J_{21}(a), J_{11}(a), J_{20}(a), J_{30}(a), J_{10}(a)$ from *Lemma* 5.2 in to the exact formula (4.2), carrying out the corresponding calculations, we finally get the asymptotic expansions (5.2).

Corollary 5.2. *Let the conditions of Theorem 5.1 be satisfied. Then the following asymptotic expansion can be written for the variance of the ergodic distribution of $X(t)$, as $a \to \infty$:*

$$\mathrm{Var}(X) = \frac{23}{144}a^2 + \frac{13\mu_{21}}{144}a + B + o(1),$$

where

$$B = \frac{m_{21}^2}{4} - \frac{m_{31}}{3} - \frac{75}{576}\mu_{21}^2 + \frac{35}{36}\mu_{31}.$$

Theorem 5.2. *Let the conditions of Theorem 4.2 be satisfied. Then the following asymptotic expansions can be written for the third and fourth moments of the ergodic distribution of $X(t)$, as $a \to \infty$:*

$$E(\overline{X}^3) = \frac{31}{60}a^3 - \left(\frac{3}{2}A_1 - \frac{31}{120}\mu_{21}\right)a^2 + \left[\left(\frac{31}{240} - \frac{7}{8}A_1\right)\mu_{21}^2 - \frac{7}{6}\mu_{31} + \frac{7}{4}A_2\right]a + o(a);$$

$$E(\overline{X}^4) = \frac{3}{5}a^4 - \left(\frac{31}{30}A_1 - \frac{3}{10}\mu_{21}\right)a^3 + \left(\frac{3}{10}\mu_{21}^2 - \frac{1}{2}A_1\mu_{21} + 3A_2\right)a^2 + o(a^2),$$

where

$$A_1 = \frac{m_{21}}{2}, A_2 = \frac{m_{21}^2}{2} - \frac{m_{31}}{3}.$$

Proof. Proof of *Theorem* 5.2 is similarly to the proof of *Theorem* 5.1. □

Remark 5.1. So, we obtained the asymptotic expansions for the first four ergodic moments of $\overline{X}(t)$. Using these moments, it is possible to calculate skewness (γ_3) and kurtosis (γ_4) of the ergodic distribution of $\overline{X}(t)$:

$$\gamma_3 = \frac{E(\overline{X} - \mu)^3}{\sigma^3}, \quad \gamma_4 = \frac{E(\overline{X} - \mu)^4}{\sigma^4} - 3, \quad \text{where } \mu = E(\overline{X}), \sigma^2 = \mathrm{Var}(\overline{X}).$$

Corollary 5.3. *Under the conditions of Theorem 5.2, the following asymptotic expansions can be written for the skewness (γ_3) and kurtosis (γ_4) of the ergodic distribution of $\overline{X}(t)$, as $a \to \infty$:*

$$\gamma_3 = 0.6056 + O\left(\frac{1}{a}\right) \quad \text{and} \quad \gamma_4 = -0.3357 + O\left(\frac{1}{a}\right).$$

6. Simulation results

Thus, the main aim of this study has been attained. But it is advisable to test the adequateness of approximated formulas to the exact ones. Fort this purpose, using the Monte Carlo experiments we can give the following simulation results. First, let us denote by $\widehat{E}(\overline{X}^k), k = \overline{1,4}$ and $\widetilde{E}(\overline{X}^k)$ the simulated and asymptotic values of the kth moment ($E(\overline{X}^k)$) of ergodic distribution of the process $\overline{X}(t)$, respectively. Moreover, we put

$$\Delta_k = |\widehat{E}(\overline{X}^k) - \widetilde{E}(\overline{X}^k)|; \quad \delta_k = \frac{\Delta_k}{\widehat{E}(X^k)} \cdot 100\%; \quad \mathrm{Ap}_k = 100\% - \delta_k, \quad k = 1,2,3.$$

In other word Δ_k, δ_k, $\mathrm{Ap}_k, k = \overline{1,4}$ are the absolute error, relative error and accuracy percentage between simulated and asymptotic values of kth ergodic moments ($E(\overline{X}^k), k = \overline{1,4}$) of the process $\overline{X}(t)$, respectively. In the Tables 1–4, the asymptotic and simulated values of $E(\overline{X}^k)$ are compared, when the variable η_1 has normal distribution with parameters (1,1). (Note that, approximated values of μ_1, μ_2, μ_3 taken from the study [9]). For the calculation of each quantity $\widehat{E}(\overline{X}^k)$, 10^8 trajectories of the process $\overline{X}(t)$ were simulated by using Monte Carlo simulation method. The asymptotic values $\widetilde{E}(\overline{X}^k)$ for $E(\overline{X}^k)$ are taken from *Theorems 5.1 and 5.2* without remainder terms, here $s = 0$. The tables with the asymptotic and simulated values of kth moment ($E(\overline{X}^k), k = \overline{1,4}$) of ergodic distribution of the process $\overline{X}(t)$ are presented as follows.

Table 1
Comparison of the asymptotic and simulated values of $E(\overline{X})$.

a	$\widehat{E}(\overline{X})$	$\widetilde{E}(\overline{X})$	Δ_1	δ_1 (%)	Ap_1 (%)
50	29.653119	29.801254400	0.148135402	0.4995609	99.50044
40	23.818905	23.961542390	0.142637388	0.5988411	99.40116
30	17.993536	18.117577920	0.124041919	0.6893693	99.31063
20	12.167070	12.262982320	0.095912317	0.7882943	99.21171
10	6.313335	6.365862175	0.052527175	0.8320036	99.16800
9	5.711445	5.768353995	0.056908995	0.9964028	99.00360
8	5.129110	5.167302104	0.038192104	0.7446146	99.25539
7	4.532171	4.561187768	0.029016768	0.6402399	99.35976
6	3.972255	3.947479764	0.024775236	0.6237071	99.37629
5	3.382124	3.321621892	0.060502108	1.7888791	98.21112

Table 2
Comparison of the asymptotic and simulated values of $E(\overline{X}^2)$.

a	$\widehat{E}(\overline{X}^2)$	$\widetilde{E}(\overline{X}^2)$	Δ_2	δ_2 (%)	Ap_2 (%)
50	1288.490552	1288.4505500	0.0400062	0.003105	99.9969
40	830.063400	830.6295770	0.5661771	0.068209	99.93179
30	472.465982	472.8086080	0.3426263	0.072519	99.92748
20	214.455051	214.9876400	0.5325885	0.248345	99.75165
10	57.314823	57.1666708	0.1481522	0.258489	99.74151
9	46.998902	46.8845739	0.1143281	0.243257	99.75674
8	37.753640	37.6024770	0.151163	0.400393	99.59961
7	29.649688	29.3203801	0.3293079	1.110662	98.88934
6	22.311508	22.0382833	0.2732247	1.224591	98.77541
5	16.222781	15.7561864	0.4665946	2.876169	97.12383

Table 3
Comparison of the asymptotic and simulated values of $E(\overline{X}^3)$.

a	$\widehat{E}(\overline{X}^3)$	$\widetilde{E}(\overline{X}^3)$	Δ_3	δ_3 (%)	Ap_3 (%)
50	67325.56881	67499.70306	174.1342555	0.2586451	99.74135
40	34837.64027	34920.18329	82.54302433	0.2369363	99.76306
30	14932.32594	14980.4531	48.1271605	0.3223018	99.6777
20	4555.035026	4580.51249	25.477464	0.5593253	99.44067
10	625.658769	620.3614558	5.297313167	0.8466777	99.15332
9	463.308736	459.1847792	4.123956775	0.8901099	99.10989
8	332.576351	328.3059984	4.2703526	1.2840217	98.71598
7	230.766167	224.6251134	6.141053642	2.6611586	97.33884
6	152.170722	145.0421241	7.1285979	4.6846054	95.31539
5	92.626459	86.45703063	6.169428375	6.6605465	93.33945

Table 4
Comparison of the asymptotic and simulated values of $E(\overline{X}^4)$.

a	$\widehat{E}(\overline{X}^4)$	$\widetilde{E}(\overline{X}^4)$	Δ_3	δ_4 (%)	Ap_4 (%)
50	3961304.609000	3962737.4000	1432.791100	0.03617	99.96383
40	1643938.562000	1644679.0100	740.445940	0.045041	99.95496
30	531992.940500	531678.4200	314.520500	0.059121	99.94088
20	109522.243000	109433.2880	88.954956	0.081221	99.91878
10	7625.999060	7641.2640	15.264936	0.20017	99.79983
9	5114.708570	5126.9421	12.233571	0.239184	99.76082
8	3279.874903	3288.2255	8.350633	0.254602	99.7454
7	2009.204611	1992.4118	16.792777	0.835792	99.16421
6	1149.358449	1121.1986	28.159761	2.450042	97.54996
5	590.531283	570.6837	19.847533	3.360962	96.63904

Remark 6.1. As seen from the presented tables, the asymptotic values provide a high accuracy to the simulated values even for small values of the parameter $a \equiv (S - s)/2$. For example, as seen from Tables 1–4 the accuracy percentage (AP) is greater than 99%, for each values of parameters $a \geqslant 9$. This indicates that the obtained approximations can safely be used for the various needs of the application.

7. Conclusion

In this study, a semi-Markovian random walk process $X(t)$ with a triangular interference of chance is considered. The asymptotic expansions are obtained for the first four moments, variance, skewness and kurtosis of the ergodic distribution of $X(t)$ as $a \to \infty$. Finally, by using Monte Carlo experiments it is shown that the given approximation formulas provide high accuracy even for small values of parameter $a \equiv (S - s)/2$. The evident and obvious forms of the asymptotic expansions allow us to observe how the initial random variables ξ_1, η_1 and ζ_1 influence to the stationary characteristics of the process $X(t)$.

It is seen from the models from the introduction section that the triangular interference of chance play an important role in practice. At the same time, the internal logic of development of the theory of semi-Markov processes also demands consideration of the triangular interference of chance.

Moreover, it is seen from Corollary 5.3, the ergodic distribution of the process $X(t)$ is skewed to the right for the large values of the parameter a, because the skewness (γ_3) is positive. At the same time, the ergodic distribution of the $X(t)$ process converges to a flattened distribution because the kurtosis (γ_4) is negative.

In the light of this information, the final form of the ergodic distribution of the process $W(t) \equiv \frac{X(t)-s}{a}$ as $a \to \infty$ can be found. Also in cases where the discrete interference of chance is not triangular, the limit form of the ergodic distribution can be obtained by using similar asymptotic methods. These limit distributions are important in the study of different real models.

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