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Statistical approximation by double complex Gauss–Weierstrass integral operators

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ABSTRACT

In this paper, we introduce the complex Gauss–Weierstrass integral operators defined on a space of analytic functions in two variables on the Cartesian product of two unit disks. Then, we study the geometric properties and statistical approximation process of our operators.

1. Introduction

The statistical approximation process of positive linear operators was first studied by Gadjiev and Orhan [1]. Recently, in the same process, the "positivity" condition of the operators has been relaxed in some sense (see [2]). Also, in recent years, in obtaining some approximation results, the concept of k-positivity for complex-valued operators (see, e.g., [3–5]) and the fuzzy positivity of fuzzy-valued operators (see [6]) have been used instead of the classical positivity of the operators. In this paper, we first introduce a sequence of double complex Gauss–Weierstrass singular integral operators and then investigate their statistical approximation properties without any type of the positivity conditions mentioned above. At the end of this paper, we also explain why we use the statistical approximation process rather than the classical one. We should note that, in a very recent paper (see [7]), we study a similar problem for the double complex Picard operators.

We first recall some concepts used in this paper. Consider the following sets:

$$D^2 := D \times D = \left\{ (z, w) \in \mathbb{C}^2 : |z| < 1 \text{ and } |w| < 1 \right\},$$
$$\bar{D}^2 := \bar{D} \times \bar{D} = \left\{ (z, w) \in \mathbb{C}^2 : |z| \le 1 \text{ and } |w| \le 1 \right\}.$$

For a complex-valued function $f : \bar{D}^2 \to \mathbb{C}$, if the univariate complex functions $f(\cdot, w)$ and $f(z, \cdot)$ (for each fixed z and $w \in D$, respectively) are analytic on D, then we say that the function $f(\cdot, \cdot)$ in two variables is analytic on D^2 (see, e.g., [8,9]). It is well-known that if a function f is analytic on D^2, then f has the following Taylor expansion:

$$f(z, w) = \sum_{k,m=0}^{\infty} a_{k,m}(f) z^k w^m, \quad (z, w) \in D^2, \tag{1.1}$$

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where the coefficients $a_{k,m}(f)$ are given by

$$a_{k,m}(f) := -\frac{1}{4\pi^2}\int_T \frac{f(p,q)}{p^{k+1}q^{m+1}}\mathrm{d}p\mathrm{d}q, \quad k,m\in\mathbb{N}_0, \tag{1.2}$$

with $T := \left\{(p,q)\in\mathbb{C}^2 : |p| = r \text{ and } |q| = \rho\right\}$ with $0 < r, \rho < 1$.

As usual, by $C(\bar{D}^2)$ we denote the space of all continuous functions on $\bar{D}^2$. Now consider also the following space:

$$A(\bar{D}^2) := \{f \in C(\bar{D}^2) : f \text{ is analytic on } D^2 \text{ with } f(0,0) = 0\}.$$

In this case, $C(\bar{D}^2)$ and $A(\bar{D}^2)$ are Banach spaces with the usual sup-norm given by $\|f\| = \sup\left\{|f(z,w)| : (z,w)\in\bar{D}^2\right\}$.

Assume now that $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers. Defining the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ by

$$\lambda_n := \frac{1}{\pi\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)}, \tag{1.3}$$

and also using the set $\mathbb{D}$ given by

$$\mathbb{D} := \left\{(s,t)\in\mathbb{R}^2 : s^2+t^2 \le \pi^2\right\},$$

we define the double complex Gauss–Weierstrass singular integral operators as follows:

$$W_n(f;z,w) = \frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}} f\left(z\mathrm{e}^{\mathrm{i}s}, w\mathrm{e}^{\mathrm{i}t}\right)\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t, \tag{1.4}$$

where $(z,w)\in\bar{D}^2$, $n\in\mathbb{N}$, $f\in A(\bar{D}^2)$, and $(\lambda_n)_{n\in\mathbb{N}}$ is given by (1.3). Then, one can easily obtain that the operators W_n preserve the constant functions.

In order to get some geometric properties of the operators W_n in (1.4) we first need the following concepts. Let $f\in C(\bar{D}^2)$. Then, the first modulus of continuity of f on $\bar{D}^2$ denoted by $\omega_1(f,\delta)_{\bar{D}^2}$, $\delta > 0$, is defined to be

$$\omega_1(f;\delta)_{\bar{D}^2} := \sup\left\{|f(z,w)-f(p,q)| : \sqrt{|z-p|^2+|w-q|^2}\le\delta,\ (z,w),(p,q)\in\bar{D}^2\right\}$$

and the second modulus of smoothness of f on $\partial(D^2)$ denoted by $\omega_2(f;\alpha)_{\partial(D^2)}$, $\alpha > 0$, is defined to be

$$\omega_2(f;\alpha)_{\partial(D^2)} := \sup\left\{\left|f\left(\mathrm{e}^{\mathrm{i}(x+s)},\mathrm{e}^{\mathrm{i}(y+t)}\right)-2f\left(\mathrm{e}^{\mathrm{i}x},\mathrm{e}^{\mathrm{i}y}\right)+f\left(\mathrm{e}^{\mathrm{i}(x-s)},\mathrm{e}^{\mathrm{i}(y-t)}\right)\right| : (x,y)\in\mathbb{R}^2 \text{ and } \sqrt{s^2+t^2}\le\alpha\right\}.$$

Then, if $\sqrt{s^2+t^2}\le\alpha$, we may write that (see [7])

$$\left|f\left(z\mathrm{e}^{\mathrm{i}s}, w\mathrm{e}^{\mathrm{i}t}\right)-2f(z,w)+f\left(z\mathrm{e}^{-\mathrm{i}s}, w\mathrm{e}^{-\mathrm{i}t}\right)\right| \le \omega_2(f;\sqrt{s^2+t^2})_{\partial(D^2)}. \tag{1.5}$$

We also get, for any $c,\alpha > 0$, that

$$\omega_2(f;c\alpha)_{\partial(D^2)} \le (1+c)^2\omega_2(f;\alpha)_{\partial(D^2)}. \tag{1.6}$$

2. Geometric properties of the operators W_n

In this section, we mainly use the idea used in [10,11]. Now let

$$B(\bar{D}^2) := \{f : \bar{D}^2 \to \mathbb{C}; f \text{ is analytic on } D^2, f(0,0) = 1 \text{ and } \mathrm{Re}[f(z,w)] > 0 \text{ for every } (z,w)\in D^2\}.$$

Then, we obtain the following main result.

Theorem 2.1. *For each fixed $n\in\mathbb{N}$, we have*

(i) $W_n(A(\bar{D}^2)) \subset A(\bar{D}^2)$,
(ii) $W_n(B(\bar{D}^2)) \subset B(\bar{D}^2)$,
(iii) $\omega_1(W_n(f);\delta)_{\bar{D}^2} \le \omega_1(f;\delta)_{\bar{D}^2}$ *for any $\delta > 0$ and for every $f\in C(\bar{D}^2)$.*

Proof. (i) Let $f\in A(\bar{D}^2)$. Then, we get $f(0,0) = 0$, and so $W_n(f;0,0) = 0$. Now we claim that $W_n(f)$ is continuous on $\bar{D}^2$. Indeed, if $(p,q),(z_m,w_m)\in\bar{D}^2$ and $\lim_m(z_m,w_m) = (p,q)$, then we get

$$\begin{aligned}
|W_n(f;z_m,w_m)-W_n(f;p,q)| &\le \frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}}\left|f\left(z_m\mathrm{e}^{\mathrm{i}s}, w_m\mathrm{e}^{\mathrm{i}t}\right)-f\left(p\mathrm{e}^{\mathrm{i}s}, q\mathrm{e}^{\mathrm{i}t}\right)\right|\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t \\
&\le \frac{\lambda_n\omega_1\left(f,\sqrt{|z_m-p|^2+|w_m-q|^2}\right)_{\bar{D}^2}}{\xi_n^2}\iint_{\mathbb{D}}\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t \\
&= \omega_1\left(f,\sqrt{|z_m-p|^2+|w_m-q|^2}\right)_{\bar{D}^2}.
\end{aligned}$$

Since $\lim_m (z_m, w_m) = (p, q)$, we may write that $\lim_m \sqrt{|z_m - p|^2 + |w_m - q|^2} = 0$, which implies that

$$\lim_m \omega_1 \left(f, \sqrt{|z_m - p|^2 + |w_m - q|^2}\right)_{\bar{D}^2} = 0$$

due to the right continuity of $\omega_1(f, \cdot)$ at zero. Hence, we get $\lim_m W_n(f; z_m, w_m) = W_n(f; p, q)$, which gives the continuity of $W_n(f)$ at the point $(p, q) \in \bar{D}^2$. Since $f \in A(\bar{D}^2)$, the function f has the Taylor expansion in (1.1) with the coefficients $a_{k,m}(f)$ in (1.2). Then, for $(z, w) \in D^2$, we get

$$f(ze^{is}, we^{it}) = \sum_{k,m=0}^{\infty} a_{k,m}(f) z^k w^m e^{i(sk+tm)}. \tag{2.1}$$

Since $|a_{k,m}(f)e^{i(sk+tm)}| = |a_{k,m}(f)|$ for every $(s, t) \in \mathbb{R}^2$, the series in (2.1) is uniformly convergent with respect to $(s, t) \in \mathbb{R}^2$. Hence, we conclude that

$$\begin{aligned} W_n(f; z, w) &= \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left(\sum_{k,m=0}^{\infty} a_{k,m}(f) z^k w^m e^{i(sk+tm)}\right) e^{-(s^2+t^2)/\xi_n^2} dsdt \\ &= \frac{\lambda_n}{\xi_n^2} \sum_{k,m=0}^{\infty} a_{k,m}(f) z^k w^m \left(\iint_{\mathbb{D}} e^{i(sk+tm)} e^{-(s^2+t^2)/\xi_n^2} dsdt\right) \\ &= \frac{\lambda_n}{\xi_n^2} \sum_{k,m=0}^{\infty} a_{k,m}(f) z^k w^m \left(\iint_{\mathbb{D}} \cos(sk + tm)\, e^{-(s^2+t^2)/\xi_n^2} dsdt\right) \\ &=: \sum_{k,m=0}^{\infty} a_{k,m}(f) \ell_n(k, m) z^k w^m, \end{aligned}$$

where, for $k, m \in \mathbb{N}_0$,

$$\begin{aligned} \ell_n(k, m) &:= \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \cos(sk + tm)\, e^{-(s^2+t^2)/\xi_n^2} dsdt \\ &= \frac{\lambda_n}{\xi_n^2} \int_0^{2\pi} \int_0^{\pi} \cos[\rho (k \cos\theta + m \sin\theta)]\, e^{-\rho^2/\xi_n^2} \rho d\rho d\theta. \end{aligned} \tag{2.2}$$

We should remark that

$$|\ell_n(k, m)| \le 1 \quad \text{for every } n \in \mathbb{N} \text{ and } k, m \in \mathbb{N}_0.$$

Therefore, for each $n \in \mathbb{N}$ and $f \in A(\bar{D}^2)$, the function $W_n(f)$ has a Taylor series expansion whose Taylor coefficients are given by

$$a_{k,m}(W_n(f)) := a_{k,m}(f) \ell_n(k, m), \quad k, m \in \mathbb{N}_0,$$

where $\ell_n(k, m)$ is given by (2.2). Combining the above facts we obtain that $W_n(f) \in A(\bar{D}^2)$. Since $f \in A(\bar{D}^2)$ was arbitrary, we immediately get that $W_n(A(\bar{D}^2)) \subset A(\bar{D}^2)$.

(ii) Now let $f \in B(\bar{D}^2)$ be fixed. As in the proof of (i), we see that $W_n(f)$ is analytic on D^2. Since $f(0, 0) = 1$, we see that $W_n(f; 0, 0) = 1$. Also, since $\mathrm{Re}[f(z, w)] > 0$ for every $(z, w) \in D^2$, we obtain that

$$\mathrm{Re}[W_n(f; z, w)] = \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \mathrm{Re}[f(ze^{is}, we^{it})] e^{-(s^2+t^2)/\xi_n^2} dsdt > 0.$$

(iii) Let $\delta > 0$ and $f \in C(\bar{D}^2)$. Assume that $(z, w), (p, q) \in \bar{D}^2$ and

$$\sqrt{|z - p|^2 + |w - q|^2} \le \delta.$$

Then, we have

$$\begin{aligned} |W_n(f; z, w) - W_n(f; p, q)| &\le \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left|f\left(ze^{is}, we^{it}\right) - f\left(pe^{is}, qe^{it}\right)\right| e^{-(s^2+t^2)/\xi_n^2} dsdt \\ &\le \omega_1 \left(f; \sqrt{|z - p|^2 + |w - q|^2}\right)_{\bar{D}^2} \\ &\le \omega_1 (f; \delta)_{\bar{D}^2}, \end{aligned}$$

which yields that

$$\omega_1(W_n(f); \delta)_{\bar{D}^2} \le \omega_1(f; \delta)_{\bar{D}^2}.$$

The proof is completed. □

3. Statistical approximation by the operators W_n

In order to obtain some statistical approximation theorems we use the concept of A-statistical convergence, where $A := [a_{jn}],\ j, n = 1, 2, \ldots,$ is any nonnegative regular summability matrix.

Recall that a matrix A is regular if $\lim_{j\to\infty}(Ax)_j = L$ whenever $\lim_{n\to\infty} x_n = L$, where the sequence $Ax = ((Ax)_j)_{j\in\mathbb{N}}$ is called the A-transform of x and defined to be

$$(Ax)_j := \sum_{n=1}^{\infty} a_{jn}x_n$$

provided that the series is convergent for each $n \in \mathbb{N}$ (see, e.g., [12]). Now, a sequence $x = (x_n)_{n\in\mathbb{N}}$ is said to be A-statistically convergent to L if, for every $\varepsilon > 0$,

$$\lim_{j\to\infty} \sum_{n:|x_n-L|\geq\varepsilon} a_{jn} = 0,$$

which is denoted by $st_A - \lim_n x_n = L$ (see [13]). If $A = C_1 = [c_{jn}]$, the Cesáro matrix of order one defined to be $c_{jn} = 1/j$ if $1 \leq n \leq j$, and $c_{jn} = 0$ otherwise, then C_1-statistical convergence coincides with the concept of statistical convergence, which was first introduced by Fast [14]. In this case, we use the notation $st - \lim$ instead of $st_{C_1} - \lim$ (see the last section for this situation). Notice that every convergent sequence is A-statistically convergent to the same value for any non-negative regular matrix A, however, its converse is not always true. Not all properties of convergent sequences hold true for A-statistical convergence (or statistical convergence). For instance, although it is well-known that a subsequence of a convergent sequence is convergent, this is not always true for A-statistical convergence. Another example is that every convergent sequence must be bounded, however it does not need to be bounded of an A-statistically convergent sequence. These important facts explain why we use the statistical convergence method rather than the usual convergence for the approximation process of our operators W_n.

In this section, we obtain the following main result.

Theorem 3.1. *Let $A := [a_{jn}],\ j, n = 1, 2, \ldots,$ be a nonnegative regular summability matrix. If $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers satisfying*

$$st_A - \lim_n \xi_n = 0, \tag{3.1}$$

then, for every $f \in A(\bar{D}^2)$, we have

$$st_A - \lim_n \|W_n(f) - f\| = 0.$$

If we take $A = C_1$ in Theorem 3.1, then we easily get the following statistical approximation result.

Corollary 3.2. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers satisfying $st - \lim_n \xi_n = 0$, then, for every $f \in A(\bar{D}^2)$, we have $st - \lim_n \|W_n(f) - f\| = 0$.*

Of course, if we choose $A = I$, the identity matrix, in Theorem 3.1, then we get the next uniform approximation result.

Corollary 3.3. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a null sequence of positive real numbers. Then, for every $f \in A(\bar{D}^2)$, the sequence $\{W_n(f)\}_{n\in\mathbb{N}}$ is uniformly convergent to f on $\bar{D}^2$.*

For proving Theorem 3.1, we need the following two lemmas.

Lemma 3.4. *For every $f \in A(\bar{D}^2)$, we have*

$$\|W_n(f) - f\| \leq \frac{M}{1 - e^{-\pi^2/\xi_n^2}} \omega_2(f, \xi_n)_{\partial(D^2)}$$

for some (finite) positive constant M independent from n.

Proof. Let $(z, w) \in \bar{D}^2$ and $f \in A(\bar{D}^2)$ be fixed. Consider the following subsets of the set $\mathbb{D}$:

$$\mathbb{D}_1 := \{(s, t) \in \mathbb{D} : s \geq 0, t \geq 0\},$$

$$\mathbb{D}_2 := \{(s, t) \in \mathbb{D} : s \leq 0, t \leq 0\},$$

$$\mathbb{D}_3 := \{(s, t) \in \mathbb{D} : s \leq 0, t \geq 0\},$$

$$\mathbb{D}_4 := \{(s, t) \in \mathbb{D} : s \geq 0, t \leq 0\}.$$

Then, we observe that

$$\begin{aligned} W_n(f;z,w)-f(z,w) &= \frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}}\left\{f\left(ze^{is},we^{it}\right)-f(z,w)\right\}e^{-(s^2+t^2)/\xi_n^2}dsdt \\ &= \frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}_1}\left\{f\left(ze^{is},we^{it}\right)-f(z,w)\right\}e^{-(s^2+t^2)/\xi_n^2}dsdt \\ &\quad+\frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}_2}\left\{f\left(ze^{is},we^{it}\right)-f(z,w)\right\}e^{-(s^2+t^2)/\xi_n^2}dsdt \\ &\quad+\frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}_3}\left\{f\left(ze^{is},we^{it}\right)-f(z,w)\right\}e^{-(s^2+t^2)/\xi_n^2}dsdt \\ &\quad+\frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}_4}\left\{f\left(ze^{is},we^{it}\right)-f(z,w)\right\}e^{-(s^2+t^2)/\xi_n^2}dsdt. \end{aligned}$$

Thus, we have

$$\begin{aligned} W_n(f;z,w)-f(z,w) &= \frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}_1}\left\{f\left(ze^{is},we^{it}\right)-2f(z,w)+f\left(ze^{-is},we^{-it}\right)\right\}e^{-(s^2+t^2)/\xi_n^2}dsdt \\ &\quad+\frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}_3}\left\{f\left(ze^{is},we^{it}\right)-2f(z,w)+f\left(ze^{-is},we^{-it}\right)\right\}e^{-(s^2+t^2)/\xi_n^2}dsdt. \end{aligned}$$

The property (1.5) implies that

$$\begin{aligned} |W_n(f;z,w)-f(z,w)| &\le \frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}_1}\omega_2\left(f,\sqrt{s^2+t^2}\right)_{\partial(D^2)}e^{-(s^2+t^2)/\xi_n^2}dsdt \\ &\quad+\frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}_3}\omega_2\left(f,\sqrt{s^2+t^2}\right)_{\partial(D^2)}e^{-(s^2+t^2)/\xi_n^2}dsdt \\ &= \frac{2\lambda_n}{\xi_n^2}\iint_{\mathbb{D}_1}\omega_2\left(f,\sqrt{s^2+t^2}\right)_{\partial(D^2)}e^{-(s^2+t^2)/\xi_n^2}dsdt \\ &= \frac{2\lambda_n}{\xi_n^2}\iint_{\mathbb{D}_1}\omega_2\left(f,\frac{\sqrt{s^2+t^2}}{\xi_n}\xi_n\right)_{\partial(D^2)}e^{-(s^2+t^2)/\xi_n^2}dsdt. \end{aligned}$$

Also using (1.6), then we obtain that

$$\begin{aligned} |W_n(f;z,w)-f(z,w)| &\le \frac{2\lambda_n\omega_2(f,\xi_n)_{\partial(D^2)}}{\xi_n^2}\iint_{\mathbb{D}_1}\left(1+\frac{\sqrt{s^2+t^2}}{\xi_n}\right)^2e^{-(s^2+t^2)/\xi_n^2}dsdt \\ &= \frac{2\lambda_n\omega_2(f,\xi_n)_{\partial(D^2)}}{\xi_n^2}\int_0^{\pi/2}\int_0^{\pi}\left(1+\frac{\rho}{\xi_n}\right)^2\rho e^{-\rho^2/\xi_n^2}d\rho d\theta \\ &= \pi\lambda_n\omega_2(f,\xi_n)_{\partial(D^2)}\int_0^{\pi}(1+u)^2ue^{-u^2}du \\ &= \frac{M}{1-e^{-\pi^2/\xi_n^2}}\omega_2(f,\xi_n)_{\partial(D^2)}, \end{aligned}$$

where

$$M=\int_0^{\pi}(1+u)^2ue^{-u^2}du<\infty.$$

Taking supremum over $(z,w)\in\bar{D}^2$ on the last inequality, the proof is completed. □

Lemma 3.5 (*See [7]*)**.** *Let $A:=[a_{jn}]$, $j,n=1,2,\ldots$, be a nonnegative regular summability matrix. If $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers satisfying (3.1), then we have, for all $f\in C(\bar{D}^2)$, that*

$$st_A-\lim_n\omega_2(f;\xi_n)_{\partial(D^2)}=0.$$

Now we are ready to prove our Theorem 3.1.

Proof of Theorem 3.1. Let $f \in A(\bar{D}^2)$. By (3.1), we easily see that

$$st_A - \lim_n \frac{1}{1 - e^{-\pi^2/\xi_n^2}} = 0.$$

Then, we may write from Lemma 3.5 that

$$st_A - \lim_n \frac{\omega_2(f;\xi_n)_{\partial(D^2)}}{1 - e^{-\pi^2/\xi_n^2}} = 0. \tag{3.2}$$

Hence, for a given $\varepsilon > 0$, it follows from Lemma 3.4 that

$$U := \{n \in \mathbb{N} : \|W_n(f) - f\| \geq \varepsilon\} \subseteq \left\{ n \in \mathbb{N} : \frac{\omega_2(f;\xi_n)_{\partial(D^2)}}{1 - e^{-\pi^2/\xi_n^2}} \geq \frac{\varepsilon}{M} \right\} =: V,$$

where M is the positive constant as in the proof of Lemma 3.4. The last inclusion gives, for every $j \in \mathbb{N}$, that

$$\sum_{n\in U} a_{jn} \leq \sum_{n\in V} a_{jn}.$$

Now letting $j \to \infty$ and then using (3.2) we obtain that

$$\lim_j \sum_{n\in U} a_{jn} = 0,$$

which implies

$$st_A - \lim_n \|W_n(f) - f\| = 0.$$

The proof is completed. □

Finally, as in [7], if we take $A = C_1$, the Cesáro matrix of order one, and define the sequence $(\xi_n)_{n\in\mathbb{N}}$ by

$$\xi_n := \begin{cases} 1, & \text{if } n = k^2,\ k = 1, 2, \ldots \\ \dfrac{1}{n}, & \text{otherwise,} \end{cases} \tag{3.3}$$

then, our statistical approximation result in Corollary 3.2 (or, Theorem 3.1) works for the operators W_n constructed with the sequence $(\xi_n)_{n\in\mathbb{N}}$ in (3.3), however the uniform approximation to a function $f \in A(\bar{D}^2)$ are impossible since $(\xi_n)_{n\in\mathbb{N}}$ is a non-convergent sequence in the usual sense. Therefore, the last example shows that our statistical approximation process used in this paper is more applicable than the classical one.

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