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# The Frictional Contact Problem of Sliding Rigid Parabolic Stamps on Graded Materials

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**Abstract.** This study presents an analytical procedure to determine the contact stress distribution at the surface of an FGM coating perfectly bonded to a homogeneous substrate. The coating is assumed to be loaded by a rigid stamp of a semi-circular or circular profile. Using Fourier transformations, the contact mechanics problems are reduced to a singular integral equation of the second kind. Singular behavior of the unknown contact stress distribution at the end points is determined by following a function theoretic method. The singular integral equation is solved numerically using an expansion-collocation technique. Main results of the study are the normal and lateral contact stress components and required contact forces as functions of material parameters. It is shown that a substrate that is softer than the FGM coating could be useful in decreasing the magnitude of the positive lateral stress at the trailing end of the contact.

**Keywords:** FGM coatings, contact stresses, circular stamp, singular integral equations.

## INTRODUCTION

The mechanics of contact problems involving functionally graded materials may generally be studied in two broad categories, namely the load transfer problems [1], and cover plates and thin films. The distinguishing feature of thin films and cover plates is that their thickness is very small compared to their other dimensions. Consequently, their bending stiffness may be neglected and they may be modeled as membranes. The load transfer problems are the more commonly known contact problems in which the loading component may be rigid or deformable and the loaded component is invariably deformable [2, 3].

In this study we will consider the load transfer problems by taking into account friction along the contacting surfaces. As some examples for the application of load transfer problems that may involve using FGMs for the purpose of improving wear resistance, one may mention such mechanical components as bearings, gears, cams, brake disks, piston rings/cylinder linings, and abradable seals in stationary gas turbines. The objective of this study is to obtain analytical benchmark solutions for examining the influence of such factors as material inhomogeneity constants, the coefficient of friction, various length parameters and the geometry of contacting solids on the critical stresses, stress concentrations and the in-plane component of the stress

on the surface that may have a bearing on the fatigue and fracture of the contacting graded materials. The sliding contact problem is formulated and solved under the assumptions of plane strain, Coulomb friction, and linear inhomogeneous elasticity. The related mixed boundary value problem is reduced to a singular integral equation of the second kind in which the contact stresses are the unknown functions. It is shown that because of friction the singularities of contact stresses are not of the standard square-root type.

## FORMULATION AND SOLUTION OF THE CONTACT MECHANICS PROBLEMS

The geometries of the contact mechanics problems considered in the present study are depicted in Fig. 1. In the problem described by Fig. 1a, an FGM coating of thickness  $h$  is perfectly bonded to a homogeneous semi-infinite substrate. The FGM coating is in sliding contact with a rigid stamp of a semi-circular profile. Normal and tangential forces transferred by the contact are given as  $P$  and  $\eta P$ , respectively,  $\eta$  being the static coefficient of friction. Fig. 1b depicts a similar type of contact mechanics problem except for the fact that there is smooth contact at both ends of the contact region. The variation in the shear modulus of the FGM coating is represented in the following form for both of the problems shown in Fig. 1:

$$\mu(y) = \mu_1 \exp(\gamma y), \quad -h < y < 0, \quad (1)$$

where  $\gamma$  is a nonhomogeneity constant. Shear modulus of the substrate is assumed to be constant and equal to  $\mu_2$ . Note that assuming  $\mu = \mu_2$  for the substrate implies that there could be a mismatch of the shear moduli at the interface  $y = -h$ . Since the Poisson's ratio does not have a significant effect on the contact stress distribution, Poisson's ratios of the coating and substrate are assumed to be constant and equal, i.e.  $\nu_1 = \nu_2 = \nu$ .

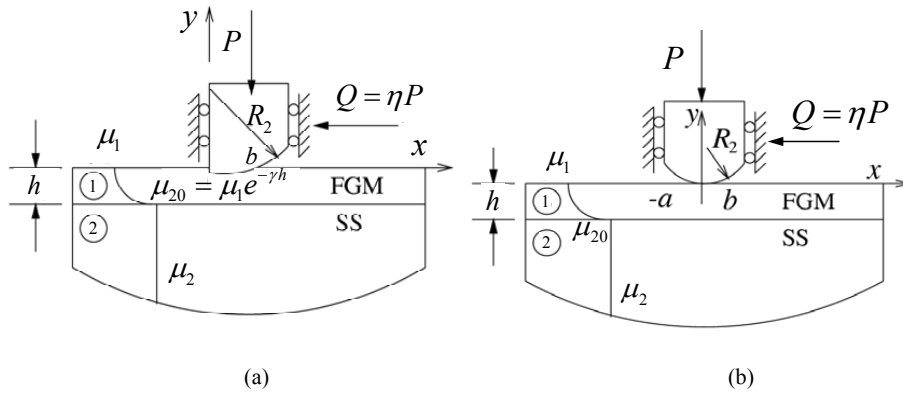
The governing partial differential equations are derived in terms of the displacement components. Assuming that both the coating and the substrate are linear elastic and that displacements are small, governing equations for the FGM coating are obtained as follows

$$(\kappa + 1) \frac{\partial^2 v_1}{\partial y^2} + (\kappa - 1) \frac{\partial^2 v_1}{\partial x^2} + 2 \frac{\partial^2 u_1}{\partial x \partial y} + \gamma(3 - \kappa) \frac{\partial u_1}{\partial x} + \gamma(\kappa + 1) \frac{\partial v_1}{\partial y} = 0, \quad -h < y < 0, \quad (2a)$$

$$(\kappa + 1) \frac{\partial^2 u_1}{\partial x^2} + (\kappa - 1) \frac{\partial^2 u_1}{\partial y^2} + 2 \frac{\partial^2 v_1}{\partial x \partial y} + \gamma(\kappa - 1) \frac{\partial u_1}{\partial y} + \gamma(\kappa - 1) \frac{\partial v_1}{\partial x} = 0, \quad -h < y < 0, \quad (2b)$$

where  $u_1$  and  $v_1$  are displacement components in  $x$  and  $y$  directions, respectively and  $\kappa = 3 - 4\nu$  for plane strain and  $\kappa = (3 - \nu)/(1 + \nu)$  for plane stress. The governing

equations for the homogeneous substrate can be obtained by substituting 0 for  $\gamma$  in Eq. (2). Additionally, equilibrium of the rigid stamp requires that,



**FIGURE 1.** An FGM coating in contact with a rigid stamp of semi-circular profile with a sharp corner (a) and circular profile with smooth ends (b).

$$\int_{-a}^b \sigma_{yy}(x, 0) dx = P, \quad (3)$$

where  $a = 0$  for the problem shown in Fig. 1a.

In the formulation of the contact mechanics problems defined above, contact stress at the surface  $y = 0$ , i.e.  $\sigma_{yy}(x, 0)$ , is used as the primary unknown function. After satisfying all the required boundary and continuity conditions analytically, the problems are reduced to a singular integral equation of the second kind. The derived integral equation has the following general form for both of the problems shown in Fig. 1:

$$\eta \omega \sigma_{yy}(x, 0) + \frac{1}{\pi} \int_{-a}^b \left[ -\frac{1}{t-x} + k_{11}(x, t) + \eta k_{12}(x, t) \right] \sigma_{yy}(t, 0) dt = f(x), \quad -a < x < b, \quad (4)$$

where  $\omega = (\kappa - 1)/(\kappa + 1)$ . The details of the derivation of Eq. (4) and the expressions for the kernels  $k_{11}(x, t)$  and  $k_{12}(x, t)$  can be found in [4]. If the radius of the semi-circular or circular stamp is much larger than the size of the contact area, the right hand side function  $f(x)$  can be approximated as follows

$$f(x) = \frac{4\mu_1}{\kappa + 1} \frac{x}{R_2}, \quad -a < x < b. \quad (5)$$

Again note that  $a = 0$  for the semi-circular stamp problem shown in Fig. 1a.

For an accurate and efficient solution of the integral equation (4) the corresponding weight function  $w(s)$  needs to be determined. By defining the complex potential

$$\Phi(z) = \frac{1}{2\pi i} \int_{-1}^1 \frac{\phi(s)}{s-z} ds, \quad (6)$$

and using the complex function theory (Muskhelishvili, 1953), from the dominant part of the integral equation (4)

$$\omega\eta\phi(r) - \frac{1}{\pi} \int_{-1}^1 \frac{\phi(s)}{s-r} ds = G(r), \quad (7)$$

the weight function of  $\phi(s)$  may be determined as

$$w(s) = (1-s)^\alpha (1+s)^\beta, \quad -1 < s < 1, \quad (8)$$

$$\alpha = \frac{\theta}{\pi} + N_0, \quad \beta = -\frac{\theta}{\pi} + M_0, \quad \theta = \arctan \frac{1}{\omega\eta}, \quad (9)$$

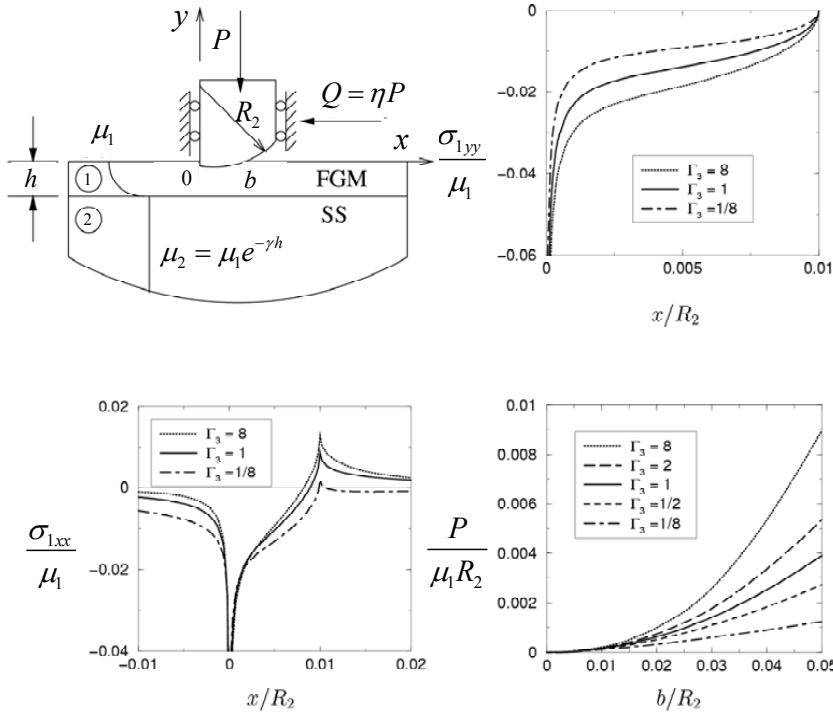
where  $N_0$  and  $M_0$  are arbitrary (positive, zero or negative) integers and are determined from the physics of the problem. In (7)  $G(r)$  represents all the bounded terms in (4). Note that the singularities  $\alpha$  and  $\beta$  are not standard square root type if the friction is present (i.e.  $\eta \neq 0$ )

The singular integral equation (4) is solved using an expansion collocation technique. First, the integral equation and the interval of definition are normalized using transformations. The bounded part of the unknown contact stress is then expanded into a series of Jacobi polynomials. Singular behavior of the weight function is determined using a function-theoretic approach. Finally, by using suitable collocation points, integral equation is converted to a linear equation system which is solved numerically to determine the contact stress distribution.

## RESULTS

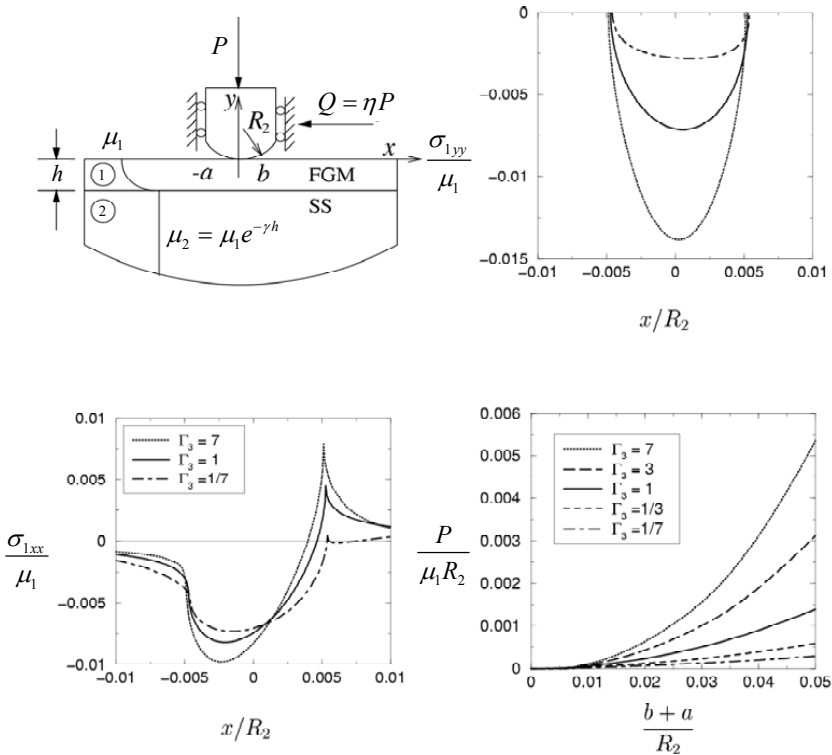
Note that in the contact mechanics problems under consideration all the field quantities can be expressed as integrals which contain the unknown contact stress  $\sigma_{yy}(x,0)$  in their integrands. Therefore, once  $\sigma_{yy}(x,0)$  is determined all the required quantities can be calculated. In addition to the  $\sigma_{yy}$  distribution at the contact surface, present study gives some results on the lateral stress component at the surface, i.e.

$\sigma_{xx}(x,0)$ , which is known to significantly influence crack initiation and growth due to sliding contact [5].



**FIGURE 2.** Stress distribution on the surface of an FGM coating loaded by a rigid semi-circular stamp for various values of the stiffness ratio,  $\Gamma_3 = \mu_2/\mu_1$ ,  $b/R_2 = 0.01$ ,  $R_2/h = 20$ ,  $\kappa = 1.8$ ,  $\eta = 0.3$ .

Sample results for semi-circular and circular stamp problems are provided in Figs. 2 and 3, respectively. The results given in Fig. 2 illustrate  $\sigma_{yy}$  and  $\sigma_{xx}$  distributions at the surface for various values of the stiffness ratio  $\Gamma_3 = \mu_2/\mu_1$ . As expected, both of the stress components are singular at the corner  $x = 0$ . A detailed analysis of the nature of singularity at this point can be found in [4]. It can also be observed that there is a positive peak for the lateral stress component at the trailing end ( $x = b$ ) of the contact. This positive peak is the main reason of crack initiation and growth in sliding contact problems. Note that in both of the contact mechanics problems considered in Fig. 1, the size of the contact zone ( $b + a$ ) is dependent upon the externally applied force  $P$ . Fig. 2 also shows the relationship between  $P$  and  $b$  for various values of the stiffness ratio. As can be seen in the figure, as the ratio  $\Gamma_3$  is increased there is also an increase in the required contact force for a given value of  $b$ . Some results on the contact stress distributions and the required contact force for the circular stamp problem are given in Fig. 3. Here, we note that  $\sigma_{yy} = 0$  at the ends due to smooth



**FIGURE 3.** Stress distribution on the surface of an FGM coating loaded by a rigid circular stamp for various values of the stiffness ratio,  $\Gamma_3 = \mu_2/\mu_1$ ,  $(b+a)/R_2 = 0.01$ ,  $R_2/h = 20$ ,  $\kappa = 1.8$ ,  $\eta = 0.3$ .

contact. The behavior of the lateral stress component is similar to that observed for the semi-circular stamp.

In closing, we also point out that, for both semi-circular and circular stamp problems, there could be a significant decrease in the magnitude of the positive lateral stress at  $x = b$ , if  $\mu_2$  is decreased while  $\mu_1$  is kept constant. This implies that a substrate less stiff than the FGM coating could be useful in avoiding crack initiation due to sliding contact.

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