NEW BINOMIAL DOUBLE SUMS WITH PRODUCTS OF FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. In this paper, we consider and compute various interesting families of binomial double sums including products of the Fibonacci and Lucas numbers. These sums have nice representations in terms of again the Fibonacci and Lucas numbers.

1. INTRODUCTION

For $n > 1$, the well-known Fibonacci sequence $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}$$

with initial values $F_0 = 0$ and $F_1 = 1$.

For $n > 1$, the well-known Lucas sequence $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2}$$

with initial values $L_0 = 2$ and $L_1 = 1$.

For various properties of these sequences as well as their different generalizations, we refer to [2, 3, 12, 14].

The Binet formulæ are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where α and β are $\left(1 \pm \sqrt{5}\right)/2$.

The relationships between negatively and positively subscripted terms of these sequences are

$$F_{-n} = (-1)^{n+1} F_n \ \text{ and } L_{-n} = (-1)^n L_n.$$

For later use, we recall that for any real number x, the floor function $\lfloor x \rfloor$ gives the greatest integer less than or equal to x. The ceiling function $\lceil x \rceil$ gives the least integer greater than or equal to x.

There are many types of summation identities including the binomial or Fibonomial coefficients, the Fibonacci, Lucas, Pell and Pell-Lucas numbers

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(see [1, 10, 11, 13, 15, 16, 17, 19]). Carlitz [1] derive various binomial sums including one binomial coefficient and the Fibonacci numbers.

The authors of [7] compute the sums including one binomial coefficient and products of the Fibonacci or Lucas numbers as well as their alternating analogues of the forms

$$\sum_{i=0}^{n}\binom{n}{i}T_{k(a+bi)}T_{k(c+di)} \quad \text{and} \quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}T_{k(a+bi)}T_{k(c+di)},$$

where T_n is either generalized Fibonacci or Lucas sequences.

Kılıç et. al. [4] give general expansion formulæ for the binomial sums with the powers of Fibonacci or Lucas numbers.

The authors of [6] derive various double binomial sums. For example, they show that

$$\sum_{i,j}\binom{n-i}{j}\binom{n-j}{i} = F_{2n+2}.$$

Kılıç and Arıkan [5] compute many binomial sums including double sums and one binomial coefficient of the forms

$$\sum_{0\le i,j\le n}\binom{i+j}{i-j} = F_{2n+2} \quad \text{and} \quad \sum_{0\le i,j\le n}\binom{i}{j-i} = F_{n+3}-1.$$

Recently, Kılıç and Taşdemir [9] also consider some special families of binomial double sums including one binomial coefficient and the Fibonacci numbers of the form

$$\sum_{0\le i,j\le n}\binom{i}{j}F_{ri+tj}$$

as well as their alternating analogues

$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i}F_{ri+tj}, \quad \sum_{0\le i,j\le n}\binom{i}{j}(-1)^{j}F_{ri+tj}, \quad \sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i+j}F_{ri+tj}$$

for some integers r and t.

More recently, Taşdemir and Toska [18] compute the binomial double sums including the Lucas numbers as well as their alternating analogues. For example, they show that

$$\sum_{0\le i,j\le n}\binom{i}{j}L_{4ti+j} = \frac{1}{L_{2t+1}}\begin{cases} L_{(2t+1)n}L_{(2t+1)(n+1)} & \text{if } n \text{ is even,}\\ 5F_{(2t+1)n}F_{(2t+1)(n+1)} & \text{if } n \text{ is odd.}\end{cases}$$

In this study, we consider various binomial double sums families whose coefficients will be chosen as products of the Fibonacci or Lucas numbers with indices in linear combination of the summation indices. These sums will be again expressed via certain linear combinations of terms F_n and

L_n. Furthermore, we compute some kinds of alternating analogues of these sums whose signs are of the forms $(-1)^{i+j}$ and $(-1)^{j+r}$.

2. Binomial Double Sums with Products of Fibonacci and Lucas Numbers

We start with an auxiliary lemma from [8] before giving the results.

Lemma 1. *For any real numbers x and y such that $x(1+y) \neq 1$.*

$$\sum_{0\leq i,j\leq k} \binom{i}{j} x^i y^j = \frac{(x+xy)^{k+1}-1}{x+xy-1}.$$

For later use, we define

$$G_n := \begin{cases} L_n & \text{if } n \text{ is even,} \\ F_n & \text{if } n \text{ is odd,} \end{cases}$$

and

$$H_n := \begin{cases} F_n & \text{if } n \text{ is even,} \\ L_n & \text{if } n \text{ is odd.} \end{cases}$$

Now we are ready to give our main results.

Theorem 1. *For any integer r,*

(1)
$$\sum_{0\leq i,j\leq n} \binom{i}{j} F_{i+j+r} F_{i+j-r} = (-1)^{r+1} F_r^2 + \frac{1}{2} 5^{\lfloor \frac{n}{2} \rfloor} H_{3n}$$

(2)
$$\sum_{0\leq i,j\leq n} \binom{i}{j} F_{i+j+r} L_{i+j-r} = -1 + (-1)^r F_{2r} + \frac{1}{2} 5^{\lceil \frac{n}{2} \rceil} G_{3n}$$

(3)
$$\sum_{0\leq i,j\leq n} \binom{i}{j} L_{i+j+r} L_{i+j-r} = (-1)^r L_r^2 + \frac{1}{2} 5^{\lfloor \frac{n}{2} \rfloor + 1} H_{3n}$$

(4)
$$\sum_{0\leq i,j\leq n} \binom{i}{j} F_{i-j+r} F_{i-j-r} = -1 + (-1)^{r+1} F_r^2 + 5^{\lfloor \frac{n}{2} \rfloor} G_{n-1}$$

(5)
$$\sum_{0\leq i,j\leq n} \binom{i}{j} F_{i-j+r} L_{i-j-r} = 1 + (-1)^r F_{2r} + 5^{\lceil \frac{n}{2} \rceil} H_{n-1}$$

(6)
$$\sum_{0\le i,j\le n}\binom{i}{j}L_{i-j+r}L_{i-j-r}=-1+5(-1)^rF_r^2+5^{\lfloor \frac{n}{2}\rfloor+1}G_{n-1}$$

(7)
$$\sum_{0\le i,j\le n}\binom{i}{j}F_{i-j-r}^2=-F_{r+1}^2+5^{\lfloor \frac{n}{2}\rfloor}G_{n-2r-1}$$

(8)
$$\sum_{0\le i,j\le n}\binom{i}{j}F_{2(i-j-r)}=F_{2(r+1)}+5^{\lceil \frac{n}{2}\rceil}H_{n-2r-1}$$

(9)
$$\sum_{0\le i,j\le n}\binom{i}{j}L_{i-j-r}^2=-L_{r+1}^2+5^{\lfloor \frac{n}{2}\rfloor+1}G_{n-2r-1}$$

(10)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i+j}F_{i+j+r}F_{i-j+r}=F_{r-1}F_{r+1}+(-1)^{r+1}5^{\lfloor \frac{n}{2}\rfloor}G_{n-1}$$

(11)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i+j}F_{i+j+r}L_{i-j+r}=F_{r-1}L_{r+1}+(-1)^{r}5^{\lceil \frac{n}{2}\rceil}H_{n-1}$$

(12)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i+j}L_{i+j+r}F_{i-j+r}=L_{r-1}F_{r+1}+(-1)^{r+1}5^{\lceil \frac{n}{2}\rceil}H_{n-1}$$

(13)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i+j}L_{i+j+r}L_{i-j+r}=L_{r-1}L_{r+1}+(-1)^{r}5^{\lfloor \frac{n}{2}\rfloor+1}G_{n-1}$$

(14)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i+j}F_{i+j+r}F_{i-j-r}=(-1)^rF_{r-1}^2+(-1)^{r+1}5^{\lfloor \frac{n}{2}\rfloor}G_{n+2r-1}$$

(15)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i+j}F_{i+j+r}L_{i-j-r}=(-1)^{r+1}F_{2(r-1)}+(-1)^{r}5^{\lceil \frac{n}{2}\rceil}H_{n+2r-1}$$

(16)

$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i+j}L_{i+j+r}F_{i-j-r}=(-1)^rF_{2(r-1)}+(-1)^{r+1}5^{\lceil\frac{n}{2}\rceil}H_{n+2r-1}$$

(17)

$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i+j}L_{i+j+r}L_{i-j-r}=(-1)^{r+1}L_{r-1}^2+(-1)^r5^{\lfloor\frac{n}{2}\rfloor+1}G_{n+2r-1}$$

(18)

$$\sum_{0\le i,j\le n}\binom{i}{j}F_{j+r}F_{j-r}=-1+(-1)^{r+1}F_r^2+5^{\lfloor\frac{n}{2}\rfloor}G_{n-1}$$

(19)

$$\sum_{0\le i,j\le n}\binom{i}{j}F_{j+r}L_{j-r}=1+(-1)^rF_{2r}+5^{\lceil\frac{n}{2}\rceil}H_{n-1}$$

(20)

$$\sum_{0\le i,j\le n}\binom{i}{j}L_{j+r}L_{j-r}=-1+5(-1)^rF_r^2+5^{\lfloor\frac{n}{2}\rfloor+1}G_{n-1}$$

(21)

$$\sum_{0\le i,j\le n}\binom{i}{j}F_{j-r}^2=-F_{r+1}^2+5^{\lfloor\frac{n}{2}\rfloor}G_{n-2r-1}$$

(22)

$$\sum_{0\le i,j\le n}\binom{i}{j}F_{2(j-r)}=F_{2(r+1)}+5^{\lceil\frac{n}{2}\rceil}H_{n-2r-1}$$

(23)

$$\sum_{0\le i,j\le n}\binom{i}{j}L_{j-r}^2=-L_{r+1}^2+5^{\lfloor\frac{n}{2}\rfloor+1}G_{n-2r-1}$$

(24)

$$\sum_{0\le i,j\le n}\binom{i}{j}F_{2i-j+r}F_{2i-j-r}=(-1)^{r+1}F_r^2+\frac{1}{2}5^{\lfloor\frac{n}{2}\rfloor}H_{3n}$$

(25)

$$\sum_{0\le i,j\le n}\binom{i}{j}F_{2i-j+r}L_{2i-j-r}=-1+(-1)^rF_{2r}+\frac{1}{2}5^{\lceil\frac{n}{2}\rceil}G_{3n}$$

(26)
$$\sum_{0\le i,j\le n}\binom{i}{j}L_{2i-j+r}L_{2i-j-r}=(-1)^rL_r^2+\frac{1}{2}5^{\lfloor\frac{n}{2}\rfloor+1}H_{3n}$$
(27)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{j-r}F_{2i-j-r}F_{2i+j-r}=(-1)^rF_{r-1}F_{r+1}-5^{\lfloor\frac{n}{2}\rfloor}G_{n-1}$$
(28)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{j-r}F_{2i-j-r}L_{2i+j-r}=(-1)^{r+1}F_{r-1}L_{r+1}-5^{\lceil\frac{n}{2}\rceil}H_{n-1}$$
(29)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{j-r}L_{2i-j-r}F_{2i+j-r}=(-1)^{r+1}L_{r-1}F_{r+1}+5^{\lceil\frac{n}{2}\rceil}H_{n-1}$$
(30)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{j-r}L_{2i-j-r}L_{2i+j-r}=(-1)^rL_{r-1}L_{r+1}+5^{\lfloor\frac{n}{2}\rfloor+1}G_{n-1}$$
(31)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{j-r}F_{2i-j+r}F_{2i+j-r}=F_{r+1}^2-5^{\lfloor\frac{n}{2}\rfloor}G_{n-2r-1}$$
(32)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{j-r}F_{2i-j+r}L_{2i+j-r}=-F_{2(r+1)}-5^{\lceil\frac{n}{2}\rceil}H_{n-2r-1}$$
(33)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{j-r}L_{2i-j+r}F_{2i+j-r}=F_{2(r+1)}+5^{\lceil\frac{n}{2}\rceil}H_{n-2r-1}$$
(34)
$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{j-r}L_{2i-j+r}L_{2i+j-r}=-L_{r+1}^2+5^{\lfloor\frac{n}{2}\rfloor+1}G_{n-2r-1}$$

Proof. As showcases, we only prove the first and eighth identities. The others could be similarly proven. We start with the first one. By the Binet formula, we write
$$\sum_{0\le i,j\le n}\binom{i}{j}F_{i+j+r}F_{i+j-r}=\frac{1}{(\alpha-\beta)^2}\sum_{0\le i,j\le n}\binom{i}{j}(\alpha^{i+j+r}-\beta^{i+j+r})(\alpha^{i+j-r}-\beta^{i+j-r}),$$

which, by $\alpha-\beta=\sqrt{5}$ and $\alpha\beta=-1$ and after some rearrangements, equals

$$\frac{1}{5}\sum_{0\le i,j\le n}\binom{i}{j}\left[\alpha^{2i+2j}+\beta^{2i+2j}+(-1)^{i+j+r+1}(\alpha^{2r}+\beta^{2r})\right]$$
$$=\frac{1}{5}\left(\sum_{0\le i,j\le n}\binom{i}{j}\left(\alpha^{2i+2j}+\beta^{2i+2j}\right)+(-1)^{r+1}\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i+j}\left(\alpha^{2r}+\beta^{2r}\right)\right)$$
$$=\frac{1}{5}\left(\sum_{0\le i,j\le n}\binom{i}{j}(\alpha^2)^i(\alpha^2)^j+\sum_{0\le i,j\le n}\binom{i}{j}(\beta^2)^i(\beta^2)^j+(-1)^{r+1}\left(\alpha^{2r}+\beta^{2r}\right)\sum_{0\le i,j\le n}\binom{i}{j}(-1)^i(-1)^j\right)$$

which, by Lemma 1, equals

$$\frac{1}{5}\left(\frac{\left(\alpha^2+\alpha^4\right)^{n+1}-1}{\alpha^2+\alpha^4-1}+\frac{\left(\beta^2+\beta^4\right)^{n+1}-1}{\beta^2+\beta^4-1}+(-1)^{r+1}(\alpha^{2r}+\beta^{2r})\frac{\left(-1+1\right)^{n+1}-1}{-1+1-1}\right)$$
$$=\quad\frac{1}{5}\left(\frac{\left(\alpha^2+\alpha^4\right)^{n+1}-1}{\alpha^2+\alpha^4-1}+\frac{\left(\beta^2+\beta^4\right)^{n+1}-1}{\beta^2+\beta^4-1}+(-1)^{r+1}(\alpha^{2r}+\beta^{2r})\right).$$

From [19], we have that $L_{2n}-2(-1)^n=5F_n^2$ and since $\alpha^2+\alpha^4=\alpha^3\sqrt{5}$, $\beta^2+\beta^4=-\beta^3\sqrt{5}$, the last expression equals

$$\frac{1}{5}\left[\frac{\left(\alpha^3\sqrt{5}\right)^{n+1}-1}{\alpha^3\sqrt{5}-1}+\frac{\left(-\beta^3\sqrt{5}\right)^{n+1}-1}{-\beta^3\sqrt{5}-1}+(-1)^{r+1}(5F_r^2+2(-1)^r)\right]$$

which, since $\alpha^3=(\sqrt{5}-2)^{-1}$ and $\beta^3=-(\sqrt{5}+2)^{-1}$, equals

$$\frac{1}{5}\left(\frac{5^{\frac{n+1}{2}}\alpha^{3n+3}-1}{2\alpha^3}+\frac{(-1)^{n+1}5^{\frac{n+1}{2}}\beta^{3n+3}-1}{2\beta^3}\right)+(-1)^{r+1}F_r^2-\frac{2}{5}$$
$$=\frac{1}{10}\left(5^{\frac{n+1}{2}}\left[\alpha^{3n}-(-1)^n\beta^{3n}\right]+\alpha^3+\beta^3\right)+(-1)^{r+1}F_r^2-\frac{2}{5}$$
$$=\frac{1}{2}5^{\frac{n-1}{2}}\left[\alpha^{3n}-(-1)^n\beta^{3n}\right]+(-1)^{r+1}F_r^2$$

In order to complete the proof, now we examine on the latest expression according to the parity of n. First, if n is even, then

$$\frac{1}{2}5^{\frac{n}{2}}F_{3n}+(-1)^{r+1}F_r^2$$
$$=(-1)^{r+1}F_r^2+\frac{1}{2}5^{\left\lfloor\frac{n}{2}\right\rfloor}F_{3n}.$$

And if n is odd, then

$$\begin{aligned}&\frac{1}{2}5^{\frac{n-1}{2}}L_{3n}+(-1)^{r+1}F_r^2\\&=(-1)^{r+1}F_r^2+\frac{1}{2}5^{\lfloor\frac{n}{2}\rfloor}L_{3n}.\end{aligned}$$

Thus the claim follows.

Now we prove the eighth identity. Consider

$$\begin{aligned}\sum_{0\le i,j\le n}\binom{i}{j}F_{2(i-j-r)} &= \frac{1}{\alpha-\beta}\left[\alpha^{-2r}\sum_{0\le i,j\le n}\binom{i}{j}\alpha^{2i-2j}-\beta^{-2r}\sum_{0\le i,j\le n}\binom{i}{j}\beta^{2i-2j}\right]\\ &= \frac{1}{\alpha-\beta}\left[\alpha^{-2r}\sum_{0\le i,j\le n}\binom{i}{j}(\alpha^2)^i(\alpha^{-2})^j-\beta^{-2r}\sum_{0\le i,j\le n}\binom{i}{j}(\beta^2)^i(\beta^{-2})^j\right],\end{aligned}$$

which, by Lemma 1, and since $\alpha+\alpha^{-1}=\sqrt{5}$ and $\beta+\beta^{-1}=-\sqrt{5}$, equals

$$\begin{aligned}&\frac{1}{\alpha-\beta}\left[\alpha^{-2r}\frac{\left(\alpha^2+1\right)^{n+1}-1}{\alpha^2+1-1}-\beta^{-2r}\frac{\left(\beta^2+1\right)^{n+1}-1}{\beta^2+1-1}\right]\\&=\frac{1}{\alpha-\beta}\left[\alpha^{-2r}\frac{\left(\alpha\sqrt{5}\right)^{n+1}-1}{\alpha^2}-\beta^{-2r}\frac{\left(-\beta\sqrt{5}\right)^{n+1}-1}{\beta^2}\right]\\&=\frac{1}{\alpha-\beta}\left(\alpha^{-2r}\frac{5^{\frac{n+1}{2}}\alpha^{n+1}-1}{\alpha^2}+\beta^{-2r}\frac{(-1)^n5^{\frac{n+1}{2}}\beta^{n+1}+1}{\beta^2}\right)\\&=\frac{1}{\alpha-\beta}\left(\alpha^{-2r}[5^{\frac{n+1}{2}}\alpha^{n-1}-\beta^2]+\beta^{-2r}[(-1)^n5^{\frac{n+1}{2}}\beta^{n-1}+\alpha^2]\right)\\&=5^{\frac{n}{2}}\alpha^{n-2r-1}+(-1)^n5^{\frac{n}{2}}\beta^{n-2r-1}+\frac{-\alpha^{-2r-2}+\beta^{-2r-2}}{\alpha-\beta}\\&=5^{\frac{n}{2}}\left(\alpha^{n-2r-1}+(-1)^n\beta^{n-2r-1}\right)+F_{2(r+1)}.\end{aligned}$$

Here we consider the latest statement to complete the proof. If n is even, then it equals

$$5^{\frac{n}{2}}L_{n-2r-1}+F_{2(r+1)},$$

as expected. And if n is odd, then it equals

$$5^{\frac{n+1}{2}}F_{n-2r-1}+F_{2(r+1)},$$

as claimed. Thus the proof is complete. □

As some special cases, we note the followings from our main result with $r=0$.

Corollary 1.

$$\sum_{0\le i,j\le n}\binom{i}{j}F_{i+j}^2=\frac{1}{2}5^{\lfloor\frac{n}{2}\rfloor}H_{3n},$$

$$\sum_{0\le i,j\le n}\binom{i}{j}F_{i-j}^2=-1+5^{\lfloor\frac{n}{2}\rfloor}G_{n-1},$$

$$\sum_{0\le i,j\le n}\binom{i}{j}F_{2(i-j)}=1+5^{\lceil\frac{n}{2}\rceil}H_{n-1},$$

$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^{i+j}F_{i+j}L_{i-j}=1+5^{\lceil\frac{n}{2}\rceil}H_{n-1},$$

$$\sum_{0\le i,j\le n}\binom{i}{j}L_j^2=-1+5^{\lfloor\frac{n}{2}\rfloor+1}G_{n-1}$$

$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^jF_{2i-j}L_{2i+j}=-1-5^{\lceil\frac{n}{2}\rceil}H_{n-1}$$

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