A NONSYMMETRICAL MATRIX AND ITS FACTORIZATIONS

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ABSTRACT. We introduce an asymmetric matrix defined by q-integers. Explicit formulæ are derived for its LU-decomposition, the inverse matrices L^{-1} and U^{-1} and its inverse. The asymmetric variants of the Filbert and Lilbert matrices come out as consequences of our results for a special value of q. The approach consists of guessing the relevant quantities and proving them later by traditional means.

1. INTRODUCTION

In classical q-analysis, the q-analogue of a nonnegative integer is defined by

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k. \tag{1}$$

From the definition, it is easily seen that

$$\lim_{q\to 1}[n]_q = n.$$

The q-Pochhammer symbol, also known as q-shifted factorial, is defined as

$$(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1}),$$

with $(x;q)_0 = 1$. Especially, when $x = q$, it is called q-factorial. (For more detail we refer to [1]).

Define the generalized Fibonacci sequence $\{U_n\}$ and generalized Lucas sequence $\{V_n\}$ by

$$U_n = pU_{n-1} + U_{n-2} \quad \text{and} \quad V_n = pV_{n-1} + V_{n-2}$$

for $n > 1$, with initial values $U_0 = 0$, $U_1 = 1$, and $V_0 = 2$, $V_1 = p$, respectively.

In particular, when $p = 1$, the sequences $\{U_n\}$ and $\{V_n\}$ are reduced to the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$, respectively.

The Binet formulæ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha-\beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \tag{2}$$

and

$$V_n = \alpha^n + \beta^n = \alpha^n(1+q^n), \tag{3}$$

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where $\alpha, \beta = \left(p \mp \sqrt{\Delta}\right)/2$ with $q = \beta/\alpha = -\alpha^2$ and $\Delta = p^2+4$, so that $\alpha = \mathbf{i}q^{-1/2}$. The RHS of (2) and (3) gives us the q-forms of the generalized Fibonacci and Lucas sequences.

The Hilbert matrix $H = [H_{ij}]$ is defined by the entries

$$H_{ij} = \frac{1}{i+j-1}.$$

As an analogue of the Hilbert matrix, Richardson [7] defined and studied the Filbert matrix $F = [F_{ij}]$ with entries

$$F_{ij} = \frac{1}{F_{i+j-1}}.$$

After the Filbert matrix, several generalizations and analogues of it have been investigated and studied by several authors. For the readers convenience, we briefly summarize some of these:

- In [3], Kılıç and Prodinger studied a generalization of the Filbert matrix by defining the matrix $\left[\frac{1}{F_{i+j+r}}\right]$, where $r \geq -1$ is an integer parameter.
- After this, Prodinger [6] defined a new generalization of the generalized Filbert matrix by introducing 3 additional parameters by taking its entries as $\frac{x^i y^j}{F_{\lambda(i+j)+r}}$, where $r \geq -1$ and $\lambda > 0$ are integers.
- In another paper [4], Kılıç and Prodinger introduced the matrix G by
$$G_{ij} = \frac{1}{F_{\lambda(i+j)+r}F_{\lambda(i+j+1)+r} \dots F_{\lambda(i+j+k-1)+r}},$$
where $r \geq -1$, $k \geq 0$ and $\lambda > 0$ are integer parameters.
- Kılıç and Prodinger [5] gave four variants of the Filbert matrix, by defining the matrices P, T, Y and Z with entries
$$P_{ij} = \frac{1}{F_{\lambda i+\mu j+r}},\ T_{ij} = \frac{F_{\lambda i+\mu j+r}}{F_{\lambda i+\mu j+s}},\ Y_{ij} = \frac{1}{L_{\lambda i+\mu j+r}} \text{ and } Z_{ij} = \frac{L_{\lambda i+\mu j+r}}{L_{\lambda i+\mu j+s}},$$
respectively, where s, r, λ and μ are integer parameters such that $s \neq r$, $r, s \geq -1$ and $\lambda, \mu > 0$.
- More recently, Kılıç and Arıkan [2] studied the nonlinear generalization of the Filbert matrix with indices in geometric progression with entries
$$\frac{1}{U_{\lambda(i+r)^k+\mu(j+s)^m+c}},$$
where U_n is the nth generalized Fibonacci number and λ, μ, k and m are positive integers, r, s and c are any integers such that $\lambda\left(i+r\right)^k + \mu\left(j+s\right)^m + c > 0$ for all positive integers i and j. They also gave its Lilbert analogue.

In the works summarized above, the authors derived explicit formulæ for the LU-decomposition, their inverses, and the Cholesky factorization. They proved the claimed results by considering the q-forms of the related quantities, and then using the celebrated q-Zeilberger algorithm and/or some algebraic manipulations.

In this paper, we introduce a new matrix $A = [A_{ij}]_{i,j\geq 0}$ defined by

$$A_{ij} = \frac{1 - xq^{\lambda i - \mu j}}{1 - xq^{\lambda i + \mu j}},$$

where λ and μ are positive integers and x is a real number such that $x \neq q^{-\lambda i - \mu j}$ for all $i, j \geq 0$.

We will derive explicit formulæ for the LU-decompositions and the inverse of the matrix A in the following section. Afterwards, we will provide proofs of the these formulæ in Section 3. It is worthwhile to note that, although all the sum identities we need to prove are q-hypergeometric summations, the q-analogue of Zeilberger's algorithm does not work for general parameters λ and μ (however, it computes the specialized sums for fixed numerical values of λ and μ). In the last section, as applications, we will give some particular results related with the generalized Fibonacci and Lucas numbers as variants of Filbert and Lilbert matrices.

Throughout the paper, the size of the matrices does not really matter and one can think of an infinite matrix A and restrict it whenever necessary to the first N rows resp. columns and write A_N.

2. Main Results

In this section, we will list the LU-decomposition of the matrix A and the L^{-1}, U^{-1} matrices and the inverse matrix A^{-1}. In the following section, we will provide the proofs of these results.

Theorem 1. *For $i, j \geq 0$,*

$$L_{ij} = \frac{(xq^{\lambda j+\mu}; q^\mu)_j (q^{\lambda(i-j+1)}; q^\lambda)_j}{(xq^{\lambda i+\mu}; q^\mu)_j (q^\lambda; q^\lambda)_j},$$

and

$$U_{ij} = \begin{cases} \dfrac{1 - xq^{-\mu j}}{1 - xq^{\mu j}} & \text{if } i = 0, \\ q^{-\mu j + (\lambda+\mu)\binom{i}{2}} x^i (1 + q^{\mu j}) \dfrac{(q^{\mu(j-i+1)}; q^\mu)_i (q^\lambda; q^\lambda)_i}{(xq^{\mu j}; q^\lambda)_{i+1} (xq^{\lambda i+\mu}; q^\mu)_{i-1}} & \text{if } i > 0. \end{cases}$$

As a consequence, one can easily compute the determinant of A, since it is simply evaluated as the product of the diagonal entries of the matrix U.

Theorem 2. *For $i, j \geq 0$,*

$$L^{-1}_{ij} = (-1)^{i+j} q^{\lambda\binom{i-j}{2}} \frac{(xq^{\lambda j+\mu}; q^\mu)_{i-1} (q^{\lambda(i-j+1)}; q^\lambda)_j}{(xq^{\lambda i+\mu}; q^\mu)_{i-1} (q^\lambda; q^\lambda)_j},$$

and

$$U_{ij}^{-1}=\begin{cases}1 & \text{if } i=j=0\\ \dfrac{q^{-\lambda\binom{j}{2}}(-1)^{j+1}(xq^{\lambda j+\mu};q^{\mu})_j}{x^j(q^{\lambda};q^{\lambda})_j} & \\ \quad\times\displaystyle\sum_{t=1}^{j}\frac{q^{\mu(\binom{t+1}{2}+t-tj)}(-1)^t(1-xq^{-\mu t})(xq^{\mu t};q^{\lambda})_j}{(1-xq^{\mu t})(1-q^{2\mu t})(q^{\mu j};q^{\mu})_{j-t}(q^{\mu};q^{\mu})_{t-1}} & \text{if } j\geq 1 \text{ and } i=0\\ (-1)^{i+j}\dfrac{q^{-\lambda\binom{j}{2}+\mu(\binom{i+1}{2}+i-ij)}}{x^j(1-q^{2\mu i})}\dfrac{(xq^{\mu i};q^{\lambda})_j(xq^{\lambda j+\mu};q^{\mu})_j}{(q^{\lambda};q^{\lambda})_j(q^{\mu};q^{\mu})_{j-i}(q^{\mu};q^{\mu})_{i-1}} & \text{if } j\geq i\geq 1,\\ 0 & \text{otherwise.}\end{cases}$$

For the inverse matrix A_N^{-1} of order N we have the following result.

Theorem 3. *For $1\leq i<N$ and $0\leq j<N$,*

$$A_{ij}^{-1}=\frac{(-1)^{i+j}}{x^{N-1}}\frac{q^{\lambda\binom{j}{2}+\mu\binom{i+1}{2}-(N-2)(\lambda j+\mu i)}}{(1-xq^{\mu i+\lambda j})(1-q^{2\mu i})}\times\frac{(xq^{\lambda j+\mu};q^{\mu})_{N-1}(xq^{\mu i};q^{\lambda})_N}{(q^{\mu};q^{\mu})_{N-i-1}(q^{\lambda};q^{\lambda})_{N-j-1}(q^{\lambda};q^{\lambda})_j(q^{\mu};q^{\mu})_{i-1}}$$

and for $0\leq j<N$,

$$A_{0j}^{-1}=[j=0]+(-1)^{j+1}q^{\lambda\binom{j}{2}-\lambda(N-2)j}\frac{x^{N-1}(xq^{\lambda j+\mu};q^{\mu})_{N-1}}{(q^{\lambda};q^{\lambda})_{N-j-1}(q^{\lambda};q^{\lambda})_j}\times\sum_{t=1}^{N}\frac{1-xq^{-\mu t}}{1-xq^{\mu t}}\frac{(-1)^t q^{\mu\binom{t+1}{2}-\mu(N-2)t}}{1-q^{2\mu t}}\frac{(xq^{\mu t};q^{\lambda})_N}{(1-xq^{\mu t+\lambda j})(q^{\mu};q^{\mu})_{N-t-1}(q^{\mu};q^{\mu})_{t-1}},$$

where $[P]$ is the Iversion notation, which is 1 when P is true, and 0 otherwise.

3. Proofs

Define the following four sums:

$$S_1(K)=\sum_{d=K}^{\min(i,j)}q^{(\lambda+\mu)\binom{d}{2}}x^d(1-xq^{d(\lambda+\mu)})\frac{(q^{\lambda(i-d+1)};q^{\lambda})_d(q^{\mu(j-d+1)};q^{\mu})_{d-1}}{(xq^{\lambda i+\mu};q^{\mu})_d(xq^{\mu j};q^{\lambda})_{d+1}},$$

$$S_2(K)=\sum_{d=j}^{K}(-1)^d q^{\lambda\binom{d-j}{2}}(1-xq^{d(\lambda+\mu)})\frac{(xq^{\lambda j+\mu};q^{\mu})_{d-1}(q^{\lambda(i-d)+\lambda};q^{\lambda})_d}{(xq^{\lambda i+\mu};q^{\mu})_d(q^{\lambda};q^{\lambda})_{d-j}},$$

$$S_3(K)=\sum_{d=i}^{K}(-1)^d q^{\mu\binom{d}{2}-\mu id}(1-xq^{d(\lambda+\mu)})\frac{(xq^{\mu i};q^{\lambda})_d(q^{\mu(j-d)+\mu};q^{\mu})_d}{(xq^{\mu j};q^{\lambda})_{d+1}(q^{\mu};q^{\mu})_{d-i}},$$

and

$$S_4(K) = \sum_{d=\max(i,j)}^{K} q^{-\mu id-\lambda dj} x^{-d}(1-xq^{d(\lambda+\mu)}) \frac{(xq^{\mu id};q^{\lambda})_d (xq^{\lambda j+\mu};q^{\mu})_{d-1} (q^{\lambda(d-j+1)};q^{\lambda})_j}{(q^{\mu};q^{\mu})_{d-i} (q^{\lambda};q^{\lambda})_d}.$$

We provide the following lemmas for later use.

Lemma 4.

$$S_1(K) = x^K q^{(\lambda+\mu)\binom{K}{2}} \frac{(q^{\lambda(i-K+1)};q^{\lambda})_K (q^{\mu(j-K+1)};q^{\mu})_{K-1}}{(1-xq^{\lambda i+\mu j})(xq^{\mu j};q^{\lambda})_K (xq^{\lambda i+\mu};q^{\mu})_{K-1}}.$$

Proof. We will use the backward induction method. Let us denote the summand term by s_d for brevity.

Firstly, assume that $i \geq j$ so when $K = j$ the claim is obvious. Similarly for the case $j > i$, the initial claim is clear.

The backward induction step amounts to show that

$$\begin{aligned}
S_1(K-1) &= S_1(K) + s_{K-1} \\
&= x^K q^{(\lambda+\mu)\binom{K}{2}} \frac{(q^{\lambda(i-K+1)};q^{\lambda})_K (q^{\mu(j-K+1)};q^{\mu})_{K-1}}{(1-xq^{\lambda i+\mu j})(xq^{\mu j};q^{\lambda})_K (xq^{\lambda i+\mu};q^{\mu})_{K-1}} \\
&+ q^{(\lambda+\mu)\binom{K-1}{2}} x^{K-1}(1-xq^{(K-1)(\lambda+\mu)}) \frac{(q^{\lambda(i-K+2)};q^{\lambda})_{K-1} (q^{\mu(j-K+2)};q^{\mu})_{K-2}}{(xq^{\lambda i+\mu};q^{\mu})_{K-1} (xq^{\mu j};q^{\lambda})_K} \\
&= x^{K-1} q^{(\lambda+\mu)\binom{K-1}{2}} \frac{(q^{\lambda(i-K+2)};q^{\lambda})_{K-1} (q^{\mu(j-K+2)};q^{\mu})_{K-2}}{(1-xq^{\lambda i+\mu j})(xq^{\mu j};q^{\lambda})_K (xq^{\lambda i+\mu};q^{\mu})_{K-1}} \\
&\times (xq^{(\lambda+\mu)(K-1)}(1-q^{\lambda j-\lambda(K-1)})(1-q^{\mu j-\mu k+\mu}) + (1-xq^{\lambda i+\mu j})(1-xq^{(K-1)(\lambda+\mu)})).
\end{aligned}$$

After some simplifications, the expression in the last line can be rewritten as

$$(1-xq^{\lambda i+\mu+\mu(K-2)})(1-xq^{\mu j+\lambda(K-1)}).$$

Finally,

$$S_1(K-1) = x^{K-1} q^{(\lambda+\mu)\binom{K-1}{2}} \frac{(q^{\lambda(i-K+2)};q^{\lambda})_{K-1} (q^{\mu(j-K+2)};q^{\mu})_{K-2}}{(1-xq^{\lambda i+\mu j})(xq^{\mu j};q^{\lambda})_{K-1} (xq^{\lambda i+\mu};q^{\mu})_{K-2}}$$

which completes the proof. ☐

Lemma 5. *For $i > j$,*

$$S_2(K) = (-1)^K q^{\lambda\binom{K-j+1}{2}} \frac{(xq^{\lambda j+\mu};q^{\mu})_K (q^{\lambda(i-K)};q^{\lambda})_{K+1}}{(1-q^{\lambda(i-j)})(xq^{\lambda i+\mu};q^{\mu})_K (q^{\lambda};q^{\lambda})_{K-j}}.$$

Proof. This time we will use the usual induction method. Similarly, we denote the summand term by s_d. The initial case $K = j$ is easily verified. So, the induction step amounts to show that

$$S_2(K+1) = S_2(K) + s_{K+1}.$$

Consider

$$\begin{aligned}
S_2(K)+s_{K+1} &= (-1)^K q^{\lambda\binom{K-j+1}{2}} \frac{(xq^{\lambda j+\mu};q^\mu)_K (q^{\lambda(i-K)};q^\lambda)_{K+1}}{(1-q^{\lambda(i-j)})(xq^{\lambda i+\mu};q^\mu)_K (q^\lambda;q^\lambda)_{K-j}} \\
&\quad + (-1)^{K+1} q^{\lambda\binom{K+1-j}{2}} (1-xq^{(K+1)(\lambda+\mu)}) \frac{(xq^{\lambda j+\mu};q^\mu)_K (q^{\lambda(i-K)};q^\lambda)_{K+1}}{(xq^{\lambda i+\mu};q^\mu)_{K+1} (q^\lambda;q^\lambda)_{K+1-j}} \\
&= (-1)^{K+1} q^{\lambda\binom{K-j+1}{2}} \frac{(xq^{\lambda j+\mu};q^\mu)_K (q^{\lambda(i-K)};q^\lambda)_{K+1}}{(1-q^{\lambda(i-j)})(xq^{\lambda i+\mu};q^\mu)_{K+1} (q^\lambda;q^\lambda)_{K+1-j}} \\
&\quad \times \big((1-xq^{(K+1)(\lambda+\mu)})(1-q^{\lambda(i-j)}) - (1-xq^{\lambda i+\mu(K+1)})(1-q^{\lambda(K+1-j)})\big) \\
&= (-1)^{K+1} q^{\lambda\binom{K-j+1}{2}} \frac{(xq^{\lambda j+\mu};q^\mu)_K (q^{\lambda(i-K)};q^\lambda)_{K+1}}{(1-q^{\lambda(i-j)})(xq^{\lambda i+\mu};q^\mu)_{K+1} (q^\lambda;q^\lambda)_{K+1-j}} \\
&\quad \times q^{\lambda(K-j+1)} (1-xq^{\lambda j+\mu(K+1)})(1-q^{\lambda(i-K)-\lambda}),
\end{aligned}$$

which is equal to $S_2(K+1)$, as desired. □

We omit the proofs of the following two lemmas due to the similarities to the proof of Lemma 5.

Lemma 6. *For $j>i$,*

$$S_3(K) = (-1)^K q^{\mu\binom{K}{2}+\mu(K(1-i)-i)} \frac{(xq^{\mu i};q^\lambda)_{K+1} (q^{\mu(j-i+1)};q^\mu)_i (q^{\mu(j-K)};q^\mu)_{K-i}}{(xq^{\mu j};q^\lambda)_{K+1} (q^\mu;q^\mu)_{K-i}}.$$

Lemma 7.

$$S_4(K) = q^{-\mu K i-\lambda K j} x^{-K} \frac{(xq^{\lambda j+\mu};q^\mu)_K (xq^{\mu i};q^\lambda)_N}{(1-xq^{\mu i+\lambda j})(q^\mu;q^\mu)_{K-i} (q^\lambda;q^\lambda)_{K-j}}.$$

Now we can give the proofs of our main results.

For the LU-decomposition of the matrix A, we have to prove that

$$\sum_{0\le d\le \min\{i,j\}} L_{id}U_{dj} = A_{ij}.$$

By Lemma 4, we obtain

$$\begin{aligned}
\sum_{0\le d\le \min\{i,j\}} L_{id}U_{dj} &= \frac{1-xq^{-\mu j}}{1-xq^{\mu j}} + q^{-\mu j}(1-q^{2\mu j}) S_1(1) \\
&= \frac{1-xq^{-\mu j}}{1-xq^{\mu j}} + q^{-\mu j} \frac{(1-q^{2\mu j})(1-q^{\lambda i})}{(1-xq^{\lambda i+\mu j})(1-xq^{\mu j})} \\
&= \frac{1-xq^{\lambda i-\mu j}}{1-xq^{\lambda i+\mu j}},
\end{aligned}$$

which completes the proof.

For L and L^{-1}, it is obvious that $l_{ii}l_{ii}^{-1} = 1$. For $i > j$,

$$\sum_{j \le d \le i} L_{id} L_{dj}^{-1} = \frac{(-1)^j}{(q^\lambda; q^\lambda)_j} S_2(i),$$

which equals 0 by Lemma 5. So we conclude

$$\sum_{j \le d \le i} L_{id} L_{dj}^{-1} = [i = j],$$

as desired.

Before moving on, notice that the matrices U^{-1} and A^{-1} can be also written as follows:

$$U^{-1} = BC \text{ and } A_N^{-1} = B_N D_N,$$

where the matrix B is defined by

$$\begin{aligned} B_{00} &= 1 \text{ and } B_{0j} = -\frac{1 - xq^{-\mu j}}{1 - xq^{\mu j}} \text{ for } j > 0, \\ B_{ij} &= [i = j] \text{ for } j \ge 0 \text{ and } i \ge 1 \end{aligned}$$

and

$$\begin{aligned} &C_{00} = 1 \text{ and } C_{0j} = 0 \text{ for } j > 0 \text{ and } C_{ij} = U_{ij}^{-1} \text{ for } j \ge i \ge 1, \\ &D_{00} = 1 \text{ and } D_{0j} = 0 \text{ for } 0 < j < N, \text{ and } D_{ij} = A_{ij}^{-1} \text{ otherwise.} \end{aligned}$$

It is easily seen that the inverse matrix B^{-1} is given by

$$\begin{aligned} B_{00}^{-1} &= 1 \text{ and } B_{0j}^{-1} = \frac{1 - xq^{-\mu j}}{1 - xq^{\mu j}} \text{ for } j > 0, \\ B_{ij}^{-1} &= [i = j] \text{ for } j \ge 0 \text{ and } i \ge 1. \end{aligned}$$

In order to show that $U^{-1}U = I$, we will show the $BCU = I$. Consider the product matrix CU. The first row of this matrix is the same as the first row of the matrix U. Then for $i \ge 1$, obviously $C_{ii}U_{ii} = 1$, so when $i \ne j$ we have

$$\sum_{i \le d \le j} C_{id} U_{dj} = (-1)^i q^{-\mu j + \frac{1}{2}\mu i(i+3)} \frac{1 + q^{\mu j}}{(1 - q^{2\mu i})(q^\mu; q^\mu)_{i-1}} S_3(j) = 0,$$

which gives $CU = B^{-1}$; so the claim follows.

Finally, for the inverse matrix A_N^{-1}, we use the fact $A_N^{-1} = U_N^{-1} L_N^{-1} = B_N C_N L_N^{-1}$. The first row of the matrix $C_N L_N^{-1}$ is $[j = 0]$ for $0 \le j \ge N - 1$. For $i \ge 1$, by Lemma 7, we obtain

$$\sum_{\max\{i,j\} \le d \le N-1} C_{id} L_{dj}^{-1} = \frac{(-1)^{i+j} q^{\mu\binom{i+1}{2} + \lambda\binom{j}{2}}}{(1 - q^{2\mu j})(q^\lambda; q^\lambda)_j (q^\mu; q^\mu)_{i-1}} S_4(N - 1) = A_{ij}^{-1}.$$

So $C_N L_N^{-1} = D_N$, which completes the proof.

We have the following useful lemma to easily obtain results for the transpose of a nonsymmetric matrix. For a given sequence $\{a_n\}$, we define the diagonal matrix $D(a_i) = [d_{ij}]$ as

$$d_{ij} = \begin{cases} a_i & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 8. *Let A be a nonsingular square matrix whose LU-decomposition is known, where $L = [L_{ij}]$, $U = [U_{ij}]$, respectively. Then we have*

$$A^T = L'U',$$

where

$$L'_{ij} = \frac{U_{ji}}{U_{jj}} \quad \text{and} \quad U'_{ij} = L_{ji}U_{ii}.$$

Proof. Consider

$$A^T = U^T L^T = U^T D\Big(\frac{1}{U_{ii}}\Big) D(U_{ii}) L^T.$$

Then $L' = U^T D(\frac{1}{U_{ii}})$ and $U' = D(U_{ii})L^T$, which completes the proof. □

So by the above lemma, one can easily derive the results for the transposed matrix $A^T = \Big[\frac{1-xq^{\lambda j-\mu i}}{1-xq^{\lambda j+\mu i}}\Big]_{i,j\geq 0}$.

4. Applications

In this section, we will give some applications of our main results. For example, consider the matrix $\mathcal{F}$, defined by

$$\mathcal{F}_{ij} = \frac{U_{\lambda i-\mu j+d}}{U_{\lambda i+\mu j+d}}$$

with positive integers λ, μ and d. By (2), the entries of the matrix $\mathcal{F}$ can be rewritten as

$$\mathcal{F}_{ij} = q^{\mu j}(-1)^{\mu j}\frac{1-q^{\lambda i-\mu j+d}}{1-q^{\lambda i+\mu j+d}},$$

where $q = \beta/\alpha$. So, for $x = q^d$ and $q = \beta/\alpha$, we can write

$$\mathcal{F} = AD(q^{\mu i}(-1)^{\mu i}),$$

where $D(a_i)$ is the diagonal matrix defined as before. So one can easily derive all related results for the matrix $\mathcal{F}$ from the results of the matrix A.

Note that an interesting feature of the matrix $\mathcal{F}$ is that it includes some zero terms as entries. Especially, when $\lambda = \mu = 1$, then the entries on the dth super-diagonal are all zero.

After some manipulations and converting the factors again to generalized Fibonacci numbers, we find the LU-decomposition of the matrix $\mathcal{F}$ and L^{-1}, U^{-1} and $\mathcal{F}^{-1}$ as follows:

$$L_{ij} = \frac{\Big(\prod_{k=1}^{j} U_{\lambda j+\mu k+d}\Big)\Big(\prod_{k=1}^{j} U_{\lambda(i+1)-\lambda k}\Big)}{\Big(\prod_{k=1}^{j} U_{\lambda i+\mu k+d}\Big)\Big(\prod_{k=1}^{j} U_{\lambda k}\Big)},$$

$$U_{ij} = \begin{cases} \dfrac{U_{-\mu j+d}}{U_{\mu j+d}} & \text{if } i=0,\\ (-1)^{\mu j+(\lambda+\mu)\binom{i}{2}+di} U_{2\mu j} \dfrac{\Big(\prod_{k=1}^{i-1} U_{\mu j-\mu k}\Big)\Big(\prod_{k=1}^{i} U_{\lambda k}\Big)}{\Big(\prod_{k=1}^{i+1} U_{\mu j+\lambda(k-1)+d}\Big)\Big(\prod_{k=1}^{i-1} U_{\lambda i+\mu k+d}\Big)} & \text{if } i>0, \end{cases}$$

$$L_{ij}^{-1} = (-1)^{i+j+\lambda\binom{i-j}{2}} \frac{\Big(\prod_{k=1}^{i-1} U_{\lambda j+\mu k+d}\Big)\Big(\prod_{k=1}^{j} U_{\lambda(i+1)-\lambda k}\Big)}{\Big(\prod_{k=1}^{i-1} U_{\lambda i+\mu k+d}\Big)\Big(\prod_{k=1}^{j} U_{\lambda k}\Big)},$$

and $U_{ij}^{-1} =$

$$\begin{cases} 1 & \text{if } i=j=0,\\ (-1)^{i+j+\lambda\binom{j}{2}+dj+\mu ij+\mu\binom{i}{2}} \dfrac{1}{U_{2\mu i}} \dfrac{\Big(\prod_{k=1}^{j} U_{\mu i+\lambda(k-1)+d}\Big)\Big(\prod_{k=1}^{j} U_{\lambda j+\mu k+d}\Big)}{\Big(\prod_{k=1}^{j} U_{\lambda k}\Big)\Big(\prod_{k=1}^{j-i} U_{\mu k}\Big)\Big(\prod_{k=1}^{i-1} U_{\mu k}\Big)} & \text{if } j \ge i \ge 1, \end{cases}$$

and for $j \ge 1$

$$U_{0j}^{-1} = \frac{(-1)^{j+1+dj+\lambda\binom{j}{2}} \prod_{k=1}^{j} U_{\lambda j+\mu k+d}}{\prod_{k=1}^{j} U_{\lambda k}} \times \sum_{t=1}^{j} (-1)^{t+\mu tj+\mu\binom{t}{2}} \frac{U_{-\mu t+d}}{U_{2\mu t} U_{\mu t+d}} \frac{\prod_{k=1}^{j} U_{\mu t+\lambda(k-1)+d}}{\Big(\prod_{k=1}^{j-t} U_{\mu k}\Big)\Big(\prod_{k=1}^{t-1} U_{\mu k}\Big)},$$

and 0 otherwise.

For the inverse matrix, we have for $1 \le i < N$ and $0 \le j < N$,

$$\mathcal{F}_{ij}^{-1} = \frac{(-1)^{i+j+\lambda\binom{j}{2}+\mu\binom{i+1}{2}+N(\lambda j+\mu i)+d(N-1)}}{U_{2\mu i}} \Big(\prod_{k=1}^{i-1} U_{\mu N+\lambda j-\mu k+d}\Big) \times \frac{\Big(\prod_{k=1}^{j} U_{\mu i+\lambda(k-1)+d}\Big)\Big(\prod_{k=1}^{N-j-1} U_{\lambda N+\mu i-\lambda k+d}\Big)\Big(\prod_{k=1}^{N-i} U_{\lambda j+\mu k+d}\Big)}{\Big(\prod_{k=1}^{N-i-1} U_{\mu k}\Big)\Big(\prod_{k=1}^{N-j-1} U_{\lambda k}\Big)\Big(\prod_{k=1}^{j} U_{\lambda k}\Big)\Big(\prod_{k=1}^{i-1} U_{\mu k}\Big)}$$

and for $0 \le j < N$,

$$\mathcal{F}_{0j}^{-1} = [j=0] - \sum_{k=1}^{N-1} \frac{U_{-\mu k+d}}{U_{\mu k+d}} \mathcal{F}_{kj}^{-1}.$$

Similarly, for the Lucas analogue, we define the matrix $\mathcal{L}$ with

$$\mathcal{L}_{ij} = \frac{V_{\lambda i-\mu j+d}}{V_{\lambda i+\mu j+d}} = q^{\mu j}(-1)^{\mu j}\frac{1+q^{\lambda i-\mu j+d}}{1+q^{\lambda i+\mu j+d}}$$

with positive integers λ, μ and integer d and $q = \beta/\alpha$. By choosing $x = -q^d$ in our main results, we have the following results for the matrix $\mathcal{L}$.

$$L_{ij} = \frac{\left(\prod_{k=1}^{j} V_{\lambda j+\mu k+d}\right)\left(\prod_{k=1}^{j} U_{\lambda(i+1)-\lambda k}\right)}{\left(\prod_{k=1}^{j} V_{\lambda i+\mu k+d}\right)\left(\prod_{k=1}^{j} U_{\lambda k}\right)},$$

$$U_{ij} = \begin{cases} \dfrac{V_{-\mu j+d}}{V_{\mu j+d}} & \text{if } i = 0, \\ \Delta^i(-1)^{\mu j+(\lambda+\mu)\binom{i}{2}+d(i+1)}U_{2\mu j}\dfrac{\left(\prod_{k=1}^{i-1} U_{\mu j-\mu k}\right)\left(\prod_{k=1}^{i} U_{\lambda k}\right)}{\left(\prod_{k=1}^{i+1} V_{\mu j+\lambda(k-1)+d}\right)\left(\prod_{k=1}^{i-1} V_{\lambda i+\mu k+d}\right)} & \text{if } i > 0, \end{cases}$$

where Δ defined as before.

$$L_{ij}^{-1} = (-1)^{i+j+\lambda\binom{i-j}{2}}\frac{\left(\prod_{k=1}^{i-1} V_{\lambda j+\mu k+d}\right)\left(\prod_{k=1}^{j} U_{\lambda(i+1)-\lambda k}\right)}{\left(\prod_{k=1}^{i-1} V_{\lambda i+\mu k+d}\right)\left(\prod_{k=1}^{j} U_{\lambda k}\right)},$$

and $U_{ij}^{-1} =$

$$\begin{cases} 1 & \text{if } i = j = 0 \\ \dfrac{(-1)^{i+\lambda\binom{j}{2}+dj+\mu ij+\mu\binom{i}{2}}}{\Delta^j}\dfrac{1}{U_{2\mu i}}\dfrac{\left(\prod_{k=1}^{j} V_{\mu i+\lambda(k-1)+d}\right)\left(\prod_{k=1}^{j} V_{\lambda j+\mu k+d}\right)}{\left(\prod_{k=1}^{j} U_{\lambda k}\right)\left(\prod_{k=1}^{j-i} U_{\mu k}\right)\left(\prod_{k=1}^{i-1} U_{\mu k}\right)} & \text{if } j \geq i \geq 1, \end{cases}$$

and for $j \geq 1$

$$U_{0j}^{-1} = \frac{(-1)^{dj+\lambda\binom{j}{2}+1}\left(\prod_{k=1}^{j} V_{\lambda j+\mu k+d}\right)}{\Delta^j\left(\prod_{k=1}^{j} U_{\lambda k}\right)}$$
$$\times\sum_{t=1}^{j}(-1)^{t+\mu tj+\mu\binom{t}{2}}\frac{V_{-\mu t+d}}{U_{2\mu t}V_{\mu t+d}}\frac{\left(\prod_{k=1}^{j} V_{\mu t+\lambda(k-1)+d}\right)}{\left(\prod_{k=1}^{j-t} U_{\mu k}\right)\left(\prod_{k=1}^{t-1} U_{\mu k}\right)},$$

and 0 otherwise.

For the inverse matrix, we have for $1 \le i < N$ and $0 \le j < N$,

$$\mathcal{L}_{ij}^{-1} = \frac{(-1)^{i+j+\lambda\binom{j}{2}+\mu\binom{i+1}{2}-N(\mu i+\lambda j)+(d+1)(N-1)}}{\Delta^{N-1}U_{2\mu i}}\Big(\prod_{k=1}^{i-1} V_{\mu N+\lambda j-\mu k+d}\Big)$$
$$\times \frac{\Big(\prod_{k=1}^{j} V_{\mu i+\lambda(k-1)+d}\Big)\Big(\prod_{k=1}^{N-j-1} V_{\lambda N+\mu i-\lambda k+d}\Big)\Big(\prod_{k=1}^{N-i} V_{\lambda j+\mu k+d}\Big)}{\Big(\prod_{k=1}^{N-i-1} U_{\mu k}\Big)\Big(\prod_{k=1}^{N-j-1} U_{\lambda k}\Big)\Big(\prod_{k=1}^{j} U_{\lambda k}\Big)\Big(\prod_{k=1}^{i-1} U_{\mu k}\Big)},$$

and for $0 \le j < N$,

$$\mathcal{L}_{0j}^{-1} = [j=0] - \sum_{k=1}^{N-1} \frac{V_{-\mu k+d}}{V_{\mu k+d}}\mathcal{L}_{kj}^{-1}.$$

More specially, by choosing $x = q^d$ such that $d > 0$ is an integer and performing the limit $q \to 1$ in our main results, we obtain the related results for the matrix $\mathcal{H} = [\mathcal{H}_{ij}]_{i,j\ge 0}$ as a variant of Hilbert matrix with entries

$$\mathcal{H}_{ij} = \frac{\lambda i - \mu j + d}{\lambda i + \mu j + d}.$$

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