EVALUATION OF SUMS INVOLVING PRODUCTS OF GAUSSIAN q-BINOMIAL COEFFICIENTS WITH APPLICATIONS

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Abstract. Sums of products of two Gaussian q-binomial coefficients, are investigated, one of which includes two additional parameters, with a parametric rational weight function. By means of partial fraction decomposition, first the main theorems are proved and then some corollaries of them are derived. Then these q-binomial identities will be transformed into Fibonomial sums as consequences.

1. Introduction

Throughout this paper we use the following notations: the q-Pochhammer symbol $(z;q)_n = (1-z)(1-zq)\dots(1-zq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{z,q} = \frac{(z;q)_n}{(z;q)_k (z;q)_{n-k}}.$$

When $z = q$, we use the notations $(q)_n$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ instead of $(q;q)_n$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{q,q}$, respectively. We conveniently adopt the notation that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = 0$ if $k < 0$ or $k > n$.

Define the $\{U_n, V_n\}$ sequences as linear recurrences for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0, \quad U_1 = 1,$$
$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2, \quad V_1 = p.$$

With $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$, they admit the following expressions in the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad V_n = \alpha^n + \beta^n.$$

When $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = (1-\sqrt{5})/(1+\sqrt{5})$), the sequence $\{U_n\}$ is reduced to the Fibonacci sequence $\{F_n\}$ and the sequence $\{V_n\}$ is reduced to the Lucas sequence $\{L_n\}$.

For $n \geq k \geq 1$, we will use *generalized Fibonomial coefficients*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_U = \frac{U_1 U_2 \dots U_n}{(U_1 U_2 \dots U_k)(U_1 U_2 \dots U_{n-k})} \quad \text{with} \quad \begin{Bmatrix} n \\ 0 \end{Bmatrix}_U = \begin{Bmatrix} n \\ n \end{Bmatrix}_U = 1.$$

When $p = 1$, they reduces to the usual Fibonomial coefficient, denoted by $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_F$. For more details about the usual Fibonomial or generalized Fibonomial coefficients, their properties and interesting appearances in various places from number theory to linear algebra see [5, 6, 7, 8, 16, 17, 18].

Our approach will essentially be based on the following connection between the generalized Fibonomial and Gaussian q-binomial coefficients

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with} \quad q = \beta/\alpha \quad \text{or, equivalently} \quad \alpha = \mathbf{i}/\sqrt{q}.$$

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Furthermore, we will use *generalized Lucanomial coefficients*

$$\left\{ {n \atop k} \right\}_V = \frac{V_1 V_2 \ldots V_n}{(V_1 V_2 \ldots V_k)(V_1 V_2 \ldots V_{n-k})}$$

with $\left\{ {n \atop n} \right\}_V = 1$ where V_n is the nth generalized Lucas number.

When $V_n = L_n$, the generalized Lucanomial coefficients are reduced to the Lucanomial coefficients denoted by $\left\{ {n \atop k} \right\}_L$:

$$\left\{ {n \atop k} \right\}_L = \frac{L_1 L_2 \ldots L_n}{(L_1 L_2 \ldots L_k)(L_1 L_2 \ldots L_{n-k})}.$$

The link between the generalized Lucanomial and Gaussian q-binomial coefficients is

$$\left\{ {n \atop k} \right\}_V = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{-q} \quad \text{with} \quad q = -\alpha^{-2} \quad \text{or} \quad \alpha = \mathbf{i}/\sqrt{q}.$$

There are many kind of sums including Gaussian q-binomial coefficients with certain weight functions or generalized Fibonomial coefficients with generalized Fibonacci and Lucas numbers as coefficients (for more details, see [9, 10, 11, 12, 13, 14, 15, 16, 18]).

Marques and Trojovsky [16] provide various sums including Fibonomial coefficients, Fibonacci and Lucas numbers. For example, for positive integers m and n, they show that

$$\sum_{j=0}^{4m+2} \left\{ {4m+2 \atop j} \right\}_F (-1)^{\frac{j(j+1)}{2}} L_{2m+1-j} = -\left\{ {4m+2 \atop 4n+3} \right\}_F \frac{F_{4n+3}}{F_{2m+1}}$$

and

$$\sum_{j=0}^{4m+2} \left\{ {4m \atop j} \right\}_F (-1)^{\frac{j(j-1)}{2}} F_{n+4m-j} = \frac{1}{2} F_{2m+n} \sum_{j=0}^{4m} \left\{ {4m \atop j} \right\}_F (-1)^{\frac{j(j-1)}{2}} L_{2m-j}.$$

Quite recently, Kılıç and Prodinger [11] compute three types of sums involving products of Gaussian q-binomial coefficients. They are of the following forms: for any real number a

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} (a - q^k),$$

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} \frac{1}{q^{-k} - a}$$

and

$$\sum_{k=0}^{n} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k+1}{2}} \frac{a - q^{-k}}{b - q^{-k}}.$$

Then they present interesting applications to generalized Fibonomial and Lucanomial sums. They prove their results by the partial fraction decomposition method. Recently the method has been explored in proving various functional identities, for more details we refer to [1, 2, 3, 4].

In this paper, we shall investigate sums of products of two Gaussian q-binomial coefficients, one of which includes two additional parameters, with a parametric rational weight function. Our results will generalize the results of [11], introducing two additional parameters. By means of partial fraction decomposition, we shall first prove the main theorems and then give two corollaries from each of them. Then these q-binomial identities will be transformed into Fibonomial sums as consequences.

We mainly compute five types of the q-binomial sums with certain weight functions: For any positive integer m, and any real numbers $p,\ q,\ a$ and b,

$$\sum_{k=0}^{n}\begin{bmatrix}m+k\\n\end{bmatrix}_{p,q}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k+1}{2}-kn}\frac{1}{1-aq^k},$$

$$\sum_{k=0}^{n}\begin{bmatrix}m+k\\n\end{bmatrix}_{p,q}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}-kn}\left[a+\frac{1}{q^{-k}}\right],$$

$$\sum_{k=0}^{n}\begin{bmatrix}m+k-1\\n-1\end{bmatrix}_{p,q}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}-nk}\left[\frac{b-q^{-k}}{a-q^{-k}}\right],$$

$$\sum_{k=0}^{n}\begin{bmatrix}m-k\\n\end{bmatrix}_{p,q}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}}\frac{1}{q^{-k}-a}$$

and

$$\sum_{k=0}^{n}\begin{bmatrix}m-k\\n\end{bmatrix}_{p,q}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k+1}{2}}\left(1-aq^{rn-k+c}\right).$$

Note that when $m=n$ and $p=q$ in the above sums, three of them give us the results of [11]. By specifying the parameters, one could derive many different Fibonomial-Fibonacci-Lucas consequences.

2. The Main results

The results we will present throughout the paper will be satisfied for $a,b,p,q,w\in\mathbb{R}$, nonnegative integer n and positive integer m. We start with our first result:

Theorem 2.1. *For nonnegative integers m and n such that $m\geq n$,*

$$\sum_{k=0}^{n}\begin{bmatrix}m+k\\n\end{bmatrix}_{p,q}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k+1}{2}-kn}\frac{1}{1-aq^k}=a^n\frac{(q;q)_n}{(p;q)_n}\times\frac{\left(a^{-1}pq^{m-n};q\right)_n}{(a;q)_{n+1}}.$$

Proof. Consider the left-hand side of the claim

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}m+k\\n\end{bmatrix}_{p,q}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k+1}{2}-kn}\frac{1}{q^{-k}-a}\\
&=\sum_{k=0}^{n}\frac{(p;q)_{m+k}}{(p;q)_{m-n+k}(p;q)_n}\frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}(-1)^k q^{\binom{k+1}{2}-kn}\frac{1}{q^{-k}-a}\\
&=\frac{(q;q)_n}{(p;q)_n}\sum_{k=0}^{n}\frac{(p;q)_{m+k}}{(p;q)_{m-n+k}}\frac{1}{(q;q)_{n-k}(q;q)_k}(-1)^k q^{-\frac{1}{2}k(2n-k+1)}\frac{1}{q^{-k}-a}\\
&=\frac{(q;q)_n}{(p;q)_n}\sum_{k=0}^{n}\frac{\left(1-pq^{m-n+k}\right)\ldots\left(1-pq^{m+k-1}\right)}{(q;q)_{n-k}(q;q)_k}(-1)^k q^{-\frac{1}{2}k(2n-k+1)}\frac{1}{q^{-k}-a}\end{aligned}$$

or without the constant factors consider

$$\begin{aligned}&\sum_{k=0}^{n}\frac{q^{kn}\left(q^{-k}-pq^{m-n}\right)\ldots\left(q^{-k}-pq^{m-1}\right)}{(q;q)_{n-k}(q;q)_k}(-1)^k q^{-\frac{1}{2}k(2n-k+1)}\frac{1}{q^{-k}-a}\\
&=\sum_{k=0}^{n}\frac{\left(q^{-k}-pq^{m-n}\right)\ldots\left(q^{-k}-pq^{m-1}\right)}{(q;q)_{n-k}(q;q)_k}(-1)^k q^{\binom{k}{2}}\frac{1}{q^{-k}-a}.\end{aligned}$$

Define the functions

$$f(z) := \frac{(z-pq^{m-n})\left(z-pq^{m-n+1}\right)\dots\left(z-pq^{m-1}\right)}{(1-z)(1-zq)\dots(1-zq^n)}\frac{1}{z-a}$$

and

$$A(z) := \frac{(z-pq^{m-n})\left(z-pq^{m-n+1}\right)\dots\left(z-pq^{m-1}\right)}{(1-z)(1-zq)\dots(1-zq^n)}.$$

Thus set $f(z) = A(z)\,\frac{1}{z-a}$. Then the partial fraction expansion reads

$$f(z) = \sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right)\dots\left(q^{-k}-pq^{m-1}\right)}{(q;q)_k(q;q)_{n-k}\left(z-q^{-k}\right)}(-1)^k q^{\binom{k}{2}}\frac{1}{q^{-k}-a} + \frac{F(n,a)}{z-a}.$$

We multiply this by z and let $z\to\infty$:

$$0 = \sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right)\dots\left(q^{-k}-pq^{m-1}\right)}{(q;q)_k(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}}\frac{1}{q^{-k}-a} + F(n,a),$$

where

$$\begin{aligned}
F(n,a) &= -\frac{(z-pq^{m-n})\left(z-pq^{m-n+1}\right)\dots\left(z-pq^{m-1}\right)}{(1-z)(1-zq)\dots(1-zq^n)}\Bigg|_{z=a} \\
&= -\frac{(a-pq^{m-n})\left(a-pq^{m-n+1}\right)\dots\left(a-pq^{m-1}\right)}{(1-a)(1-aq)\dots(1-aq^n)} \\
&= -\frac{a^n\left(1-a^{-1}pq^{m-n}\right)\left(1-a^{-1}pq^{m-n+1}\right)\dots\left(1-a^{-1}pq^{m-1}\right)}{(1-a)(1-aq)\dots(1-aq^n)} \\
&= -a^n\frac{\left(a^{-1}pq^{m-n};q\right)_n}{(a;q)_{n+1}}.
\end{aligned}$$

Thus we write

$$\sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right)\left(q^{-k}-pq^{m-n+1}\right)\dots\left(q^{-k}-pq^{m-1}\right)}{(q;q)_k(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}}\frac{1}{q^{-k}-a} = \frac{a^n\left(a^{-1}pq^{m-n};q\right)_n}{(a;q)_{n+1}}$$

or

$$\sum_{k=0}^{n} \frac{(p;q)_{m+k}\,(q;q)_n}{(p;q)_n\,(p;q)_{m-n+k}\,(q)_k(q)_{n-k}}(-1)^k q^{\binom{k}{2}}q^{k-kn}\frac{1}{1-aq^k} = a^n\frac{(q;q)_n}{(p;q)_n}\times\frac{\left(a^{-1}pq^{m-n};q\right)_n}{(a;q)_{n+1}},$$

which, after some rearrangements, gives us

$$\sum_{k=0}^{n} \begin{bmatrix} m+k \\ n \end{bmatrix}_{p,q}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}-kn}\frac{1}{1-aq^k} = a^n\frac{(q;q)_n}{(p;q)_n}\times\frac{\left(a^{-1}pq^{m-n};q\right)_n}{(a;q)_{n+1}},$$

as claimed. □

As consequences of Theorem 2.1, we give the following results:

Corollary 2.1.1. *For nonnegative integers m and n such that $m\ge n$,*

$$\begin{aligned}
&\sum_{k=0}^{n} \begin{bmatrix} m+k \\ n \end{bmatrix}_{-q}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}-kn}\frac{1}{1+wq^{rn+c-k-1}} \\
&= (-1)^n q^{-\binom{n+1}{2}}\frac{(q;q)_n}{(-q;q)_n}\times\frac{\left(wq^{m+(r-1)n+c};q\right)_n}{\left(-wq^{n(r-1)+c-1};q\right)_{n+1}}.
\end{aligned}$$

Proof. It is enough to take $p = -q$ and $a = -w^{-1}q^{-rn-c+1}$ in Theorem 2.1. Then the claimed result follows after some rearrangements. ☐

Corollary 2.1.2. *For nonnegative integers m and n such that $m \geq n$,*

$$\sum_{k=0}^{n} \begin{bmatrix} m+k \\ n \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \, q^{\binom{k}{2}-kn} \frac{1}{1-wq^{rn-k+c}}$$
$$= (-1)^n \, \frac{q^{-\binom{n+1}{2}}}{1-wq^{(r-1)n+c}} \begin{bmatrix} rn+c \\ n \end{bmatrix}_{wq,q}^{-1} \begin{bmatrix} m+rn+c \\ n \end{bmatrix}_{wq,q}.$$

Proof. It is enough to take $p = q$ and $a = w^{-1}q^{-rn-c}$ in Theorem 2.1. Then the claimed result follows after some rearrangements with the definitions of the Gaussian q-binomial coefficient and the q-Pochhammer notation. ☐

Theorem 2.2. *For nonnegative integers m and n such that $m \geq n$,*

$$\sum_{k=0}^{n} \begin{bmatrix} m+k \\ n \end{bmatrix}_{p,q} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}-kn} \left(a + \frac{1}{q^{-k}}\right) = (-1)^n q^{-\binom{n+1}{2}} \left[a + p^n q^{mn}\right] \frac{(q;q)_n}{(p;q)_n}.$$

Proof. Consider the left-hand side of the claim

$$\sum_{k=0}^{n} \begin{bmatrix} m+k \\ n \end{bmatrix}_{p,q} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \, q^{\binom{k}{2}-kn} \left(a + \frac{1}{q^{-k}}\right)$$
$$= \frac{(q;q)_n}{(p;q)_n} \sum_{k=0}^{n} \frac{(p;q)_{m+k}}{(p;q)_{m-n+k}} \frac{1}{(q;q)_{n-k} \, (q;q)_k} (-1)^k \, q^{\binom{k}{2}-kn} \left(a + \frac{1}{q^{-k}}\right)$$
$$= \frac{(q;q)_n}{(p;q)_n} \sum_{k=0}^{n} \frac{\left(1-pq^{m-n+k}\right) \dots \left(1-pq^{m+k-1}\right)}{(q;q)_{n-k} \, (q;q)_k} (-1)^k \, q^{\binom{k}{2}-kn} \left(a + \frac{1}{q^{-k}}\right)$$
$$= \frac{(q;q)_n}{(p;q)_n} \sum_{k=0}^{n} \frac{q^{kn} \left(q^{-k}-pq^{m-n}\right) \dots \left(q^{-k}-pq^{m-1}\right)}{(q;q)_{n-k} \, (q;q)_k} (-1)^k \, q^{\binom{k}{2}-kn} \left(a + \frac{1}{q^{-k}}\right)$$
$$= \frac{(q;q)_n}{(p;q)_n} \sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right) \dots \left(q^{-k}-pq^{m-1}\right)}{(q;q)_{n-k} \, (q;q)_k} (-1)^k \, q^{\binom{k}{2}} \left(a + \frac{1}{q^{-k}}\right)$$

or without the constant it takes the form

$$\sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right) \dots \left(q^{-k}-pq^{m-1}\right)}{(q;q)_{n-k} \, (q;q)_k} (-1)^k \, q^{\binom{k}{2}} \left(a + \frac{1}{q^{-k}}\right).$$

Now define the functions

$$f(z) := \frac{\left(z-pq^{m-n}\right)\left(z-pq^{m-n+1}\right) \dots \left(z-pq^{m-1}\right)}{(1-z)(1-zq)\dots(1-zq^n)} \left(a + \frac{1}{z}\right)$$

and

$$A(z) := \frac{\left(z-pq^{m-n}\right)\left(z-pq^{m-n+1}\right) \dots \left(z-pq^{m-1}\right)}{(1-z)(1-zq)\dots(1-zq^n)}.$$

Thus we see that $f(z) = A(z)\left(a + \frac{1}{z}\right)$. Then the partial fraction expansion reads

$$f(z) = \sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right) \dots \left(q^{-k}-pq^{m-1}\right)}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k}{2}} \left(a + \frac{1}{q^{-k}}\right) + \frac{C}{z}.$$

We multiply this by z and let $z \to \infty$:

$$a(-1)^n q^{-\binom{n+1}{2}} = \sum_{k=0}^{n} \frac{\left(q^{-k} - pq^{m-n}\right) \dots \left(q^{-k} - pq^{m-1}\right)}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k}{2}} \Big(a + \frac{1}{q^{-k}}\Big) + C,$$

where

$$C = -(-1)^n p^n q^{(m-n)n + \binom{n}{2}}.$$

Thus we write

$$\begin{aligned}&\sum_{k=0}^{n} \frac{\left(q^{-k} - pq^{m-n}\right) \dots \left(q^{-k} - pq^{m-1}\right)}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k}{2}} \Big(a + \frac{1}{q^{-k}}\Big) \\ &= a(-1)^n q^{-\binom{n+1}{2}} + (-1)^n p^n q^{(m-n)n + \binom{n}{2}}\end{aligned}$$

or

$$\begin{aligned}&\sum_{k=0}^{n} \frac{\left(1 - pq^{m-n+k}\right) \dots \left(1 - pq^{m+k-1}\right)}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k}{2} - kn} \Big(a + \frac{1}{q^{-k}}\Big) \\ &= (-1)^n \Big[a q^{-\binom{n+1}{2}} + p^n q^{(m-n)n + \binom{n}{2}}\Big]\end{aligned}$$

or

$$\begin{aligned}&\sum_{k=0}^{n} \begin{bmatrix} m+k \\ n \end{bmatrix}_{p,q} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2} - kn} \Big(a + \frac{1}{q^{-k}}\Big) \\ &= (-1)^n q^{-\binom{n+1}{2}} \frac{(q;q)_n}{(p;q)_n} \left[a + p^n q^{mn}\right],\end{aligned}$$

as claimed. □

As consequences of Theorem 2.2, by taking special values of p and a, we give the following corollaries:

Corollary 2.2.1. *For nonnegative integers m and n such that $m \geq n$,*

$$\sum_{k=0}^{n} \begin{bmatrix} m+k \\ n \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-\frac{1}{2}k(2n-k+1)} \left(1 + wq^{rn+c+k}\right) = (-1)^n q^{-\binom{n+1}{2}} \Big(1 + wq^{c+n(1+m+r)}\Big).$$

Corollary 2.2.2. *For nonnegative integers m and n such that $m \geq n$,*

$$\begin{aligned}&\sum_{k=0}^{n} \begin{bmatrix} m+k \\ n \end{bmatrix}_{-q} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(1 - wq^{k+rn+c}\right) (-1)^k q^{\binom{k}{2} - kn} \\ &= (-1)^n q^{-\binom{n+1}{2}} \Big[1 - (-1)^n wq^{n(m+r+1)+c}\Big] \frac{(q;q)_n}{(-q;q)_n}.\end{aligned}$$

Theorem 2.3. *For positive integers m and n such that $m \geq n$,*

$$\sum_{k=0}^{n} \begin{bmatrix} m+k-1 \\ n-1 \end{bmatrix}_{p,q} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2} - nk} \frac{q^{-k} - b}{q^{-k} - a} = a^{n-1}(a-b) \frac{(q;q)_n}{(p;q)_{n-1}} \times \frac{(q^{m-n}p/a;q)_{n-1}}{(a;q)_{n+1}}.$$

Proof. Rewrite the LHS of the claim as

$$\begin{aligned}&\sum_{k=0}^{n} \begin{bmatrix} m+k-1 \\ n-1 \end{bmatrix}_{p,q} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2} - nk} \frac{q^{-k} - b}{q^{-k} - a} \\ &= \sum_{k=0}^{n} \frac{(p;q)_{m+k-1}}{(p;q)_{n-1} (p;q)_{m-n+k}} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k}{2} - nk} \frac{q^{-k} - b}{q^{-k} - a}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(q;q)_n}{(p;q)_{n-1}} \sum_{k=0}^{n} \frac{(p;q)_{m+k-1}}{(p;q)_{m-n+k}} \frac{1}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k}{2}-nk} \frac{q^{-k}-b}{q^{-k}-a} \\
&= \frac{(q;q)_n}{(p;q)_{n-1}} \sum_{k=0}^{n} \frac{\left(1-pq^{m-n+k}\right)\dots\left(1-pq^{m+k-2}\right)}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k}{2}-nk} \frac{q^{-k}-b}{q^{-k}-a}.
\end{aligned}
$$

Without constant factors, consider

$$
\begin{aligned}
&\sum_{k=0}^{n} \frac{q^{kn}\left(q^{-k}-pq^{m-n}\right)\dots\left(q^{-k}-pq^{m-2}\right)}{(q)_k (q)_{n-k}} (-1)^k q^{\binom{k}{2}-nk} \frac{q^{-k}-b}{q^{-k}-a} \\
&= \sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right)\dots\left(q^{-k}-pq^{m-2}\right)}{(q)_k (q)_{n-k}} (-1)^k q^{\binom{k}{2}} \frac{q^{-k}-b}{q^{-k}-a}.
\end{aligned}
$$

Now define the function

$$
A(z) := \frac{\left(z-pq^{m-n}\right)\dots\left(z-pq^{m-2}\right)}{(1-z)(1-zq)\dots(1-zq^n)} \frac{z-b}{z-a}.
$$

The partial fraction decomposition of $A(z)$ takes the form:

$$
A(z) = \sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right)\dots\left(q^{-k}-pq^{m-2}\right)}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k+1}{2}} \frac{q^{-k}-b}{q^{-k}-a} \frac{1}{1-zq^k} + \frac{F(n,a)}{z-a}.
$$

Now we multiply this relation by z and let $z \to \infty$ and obtain

$$
0 = \lim_{z\to\infty} \left(\sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right)\dots\left(q^{-k}-pq^{m-2}\right)}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k+1}{2}} \frac{q^{-k}-b}{q^{-k}-a} \frac{z}{(1-zq^k)} + F(n,a) \frac{z}{z-a} \right),
$$

which gives us

$$
0 = \sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right)\dots\left(q^{-k}-pq^{m-2}\right)}{(q;q)_k (q;q)_{n-k}} (-1)^{k-1} q^{\binom{k}{2}} \frac{q^{-k}-b}{q^{-k}-a} + F(n,a)
$$

or

$$
\sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right)\dots\left(q^{-k}-pq^{m-2}\right)}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k}{2}} \frac{q^{-k}-b}{q^{-k}-a} = F(n,a),
$$

where

$$
\begin{aligned}
F(n,a) &= \left((z-b) \frac{\left(z-pq^{m-n}\right)\left(z-pq^{m-n+1}\right)\dots\left(z-pq^{m-2}\right)}{(1-z)(1-zq)\dots(1-zq^n)}\right)\Bigg|_{z=a} \\
&= (a-b) \frac{\left(a-pq^{m-n}\right)\left(a-pq^{m-n+1}\right)\dots\left(a-pq^{m-2}\right)}{(1-a)(1-aq)\dots(1-aq^n)} \\
&= a(-1)^n q^{-\binom{n+1}{2}} + a^{n-1}(a-b) \frac{\left(q^{m-n}p/a;q\right)_{n-2}}{(a;q)_{n+1}}.
\end{aligned}
$$

Thus we get

$$
\sum_{k=0}^{n} \frac{\left(q^{-k}-pq^{m-n}\right)\dots\left(q^{-k}-pq^{m-2}\right)}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k}{2}} \frac{q^{-k}-b}{q^{-k}-a} = a^{n-1}(a-b) \frac{\left(q^{m-n}p/a;q\right)_{n-2}}{(a;q)_{n+1}}
$$

or

$$
\sum_{k=0}^{n} \frac{q^{-k(n-1)}\left(1-pq^{m-n+k}\right)\dots\left(1-pq^{m+k-2}\right)}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k}{2}} \frac{q^{-k}-b}{q^{-k}-a} = a^{n-1}(a-b) \frac{\left(q^{m-n}p/a;q\right)_{n-2}}{(a;q)_{n+1}}
$$

or

$$\frac{(q;q)_n}{(p;q)_{n-1}}\sum_{k=0}^{n}\frac{\left(1-pq^{m-n+k}\right)\dots\left(1-pq^{m+k-2}\right)}{(q;q)_k(q;q)_{n-k}}(-1)^k q^{-(n-1)k+\binom{k}{2}}\frac{q^{-k}-b}{q^{-k}-a}$$
$$=a^{n-1}\left(a-b\right)\frac{\left(q^{m-n}p/a;q\right)_{n-2}}{(a;q)_{n+1}}$$

or

$$\sum_{k=0}^{n}\begin{bmatrix}m+k-1\\ n-1\end{bmatrix}_{p,q}\begin{bmatrix}n\\ k\end{bmatrix}_q(-1)^k q^{-(n-1)k+\binom{k}{2}}\frac{q^{-k}-b}{q^{-k}-a}=a^{n-1}\left(a-b\right)\frac{(q;q)_n}{(p;q)_{n-1}}\times\frac{\left(q^{m-n}p/a;q\right)_{n-1}}{(a;q)_{n+1}},$$

as wanted. □

As consequences of Theorem 2.3, we give the following corollaries by taking special values of p and a:

Corollary 2.3.1. *For positive integers m and n such that $m \geq n$,*

$$\sum_{k=0}^{n}\begin{bmatrix}m+k-1\\ n-1\end{bmatrix}_{q}\begin{bmatrix}n\\ k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}-nk+k}\frac{1-q^k}{1-q^{n+k}}=-q^{n(n-1)}\left(1-q^n\right)^2\frac{\left(q^{m-2n+1};q\right)_{n-1}}{(q^n;q)_{n+1}}.$$

Corollary 2.3.2. *For positive integers m and n such that $m \geq n$,*

$$\sum_{k=0}^{n}\begin{bmatrix}m+k-1\\ n-1\end{bmatrix}_{-q}\begin{bmatrix}n\\ k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}-nk}\frac{1-q^{n+k}}{1+q^k}=\left(1-q^n\right)\frac{(q;q)_n}{(-q;q)_{n-1}}\times\frac{\left(q^{m-n+1};q\right)_{n-1}}{(-1;q)_{n+1}}.$$

Finally we give the following theorems without proof which could be similarly done.

Theorem 2.4. *For nonnegative integers m and n such that $m \geq 2n$,*

$$\sum_{k=0}^{n}\begin{bmatrix}m-k\\ n\end{bmatrix}_{p,q}\begin{bmatrix}n\\ k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}}\frac{1}{q^{-k}-a}=\frac{(q;q)_n}{(p;q)_n}\times\frac{\left(apq^{m-n};q\right)_n}{(a;q)_{n+1}}.$$

As consequences of Theorem 2.4, we give the following corollaries:

Corollary 2.4.1. *For nonnegative integers m and n such that $m \geq 2n$,*

$$\sum_{k=0}^{n}\begin{bmatrix}m-k\\ n\end{bmatrix}_{q}\begin{bmatrix}n\\ k\end{bmatrix}_q(-1)^k q^{\binom{k+1}{2}}\frac{1}{1-wq^{k+rn+c-1}}$$
$$=\frac{1}{1-wq^{(r+1)n+c-1}}\begin{bmatrix}rn+m+c\\ n\end{bmatrix}_{w,q}\begin{bmatrix}(r+1)\,n+c-1\\ n\end{bmatrix}_{w,q}^{-1}.$$

Corollary 2.4.2. *For nonnegative integers m and n such that $m \geq 2n$,*

$$\sum_{k=0}^{n}\begin{bmatrix}m-k\\ n\end{bmatrix}_{-q}\begin{bmatrix}n\\ k\end{bmatrix}_q(-1)^k q^{\binom{k+1}{2}}\frac{1}{1+wq^{k+rn+c}}=\frac{(q;q)_n}{(-q;q)_n}\times\frac{\left(wq^{(r-1)n+m+c+1};q\right)_n}{\left(-wq^{rn+c};q\right)_{n+1}}.$$

Now we give our main last result.

Theorem 2.5. *For nonnegative integers m and n such that $m \geq 2n$,*

$$\sum_{k=0}^{n}\begin{bmatrix}m-k\\ n\end{bmatrix}_{p,q}\begin{bmatrix}n\\ k\end{bmatrix}_q(-1)^k q^{\binom{k+1}{2}}\left(a+q^{-k}\right)=\left[a+p^n q^{n(m-n-1)}\right]\frac{(q;q)_n}{(p;q)_n}.$$

As consequences of Theorem 2.5, we have the following corollaries:

Corollary 2.5.1. *For nonnegative integers m and n such that $m \geq 2n$,*

$$\sum_{k=0}^{n} \begin{bmatrix} m-k \\ n \end{bmatrix}_{-q} \begin{bmatrix} n \\ k \end{bmatrix}_{q} (-1)^{k} q^{\binom{k+1}{2}} \left(1-(-1)^{n} wq^{rn-k+c}\right) = \left[1-wq^{(m-n+r)n+c}\right] \frac{(q;q)_n}{(-q;q)_n}.$$

Corollary 2.5.2. *For nonnegative integers m and n such that $m \geq 2n$,*

$$\sum_{k=0}^{n} \begin{bmatrix} m-k \\ n \end{bmatrix}_{q} \begin{bmatrix} n \\ k \end{bmatrix}_{q} (-1)^{k} q^{\binom{k+1}{2}} \left(1-wq^{rn-k+c}\right) = \left[1-wq^{(m-n+r)n+c}\right].$$

3. Applications

In this section, we shall give applications of our results as the generalized Fibonomial sums identities.

(1) For positive even n, positive m and all integers r and c such that $m-(r+1)n \geq c$ and $rn+c \geq 1$,

$$\sum_{k=0}^{n} \begin{Bmatrix} m+k \\ n \end{Bmatrix}_{U} \begin{Bmatrix} n \\ k \end{Bmatrix}_{U} (-1)^{\binom{k}{2}} \frac{1}{U_{k+rn+c}} = \frac{1}{U_{(r+1)n+c}} \begin{Bmatrix} m-rn-c \\ n \end{Bmatrix}_{U} \begin{Bmatrix} (r+1)n+c-1 \\ n \end{Bmatrix}_{U}^{-1},$$

$$\sum_{k=0}^{n} \begin{Bmatrix} m+k \\ n \end{Bmatrix}_{U} \begin{Bmatrix} n \\ k \end{Bmatrix}_{U} (-1)^{\binom{k}{2}} \frac{1}{V_{k+rn+c}} = \frac{1}{V_{(r+1)n+c}} \begin{Bmatrix} m-rn-c \\ n \end{Bmatrix}_{V} \begin{Bmatrix} (r+1)n+c-1 \\ n \end{Bmatrix}_{V}^{-1}.$$

(2) For positive even n, positive m, and all integers r and c such that $rn+c \geq 1$,

$$\sum_{k=0}^{n} \begin{Bmatrix} m+k \\ n \end{Bmatrix}_{V} \begin{Bmatrix} n \\ k \end{Bmatrix}_{U} (-1)^{\binom{k}{2}} \frac{1}{U_{k+rn+c}} = \frac{1}{U_{(r+1)n+c}} \begin{Bmatrix} (r+1)n+c-1 \\ n \end{Bmatrix}_{U}^{-1} \begin{Bmatrix} m-rn-c \\ n \end{Bmatrix}_{V},$$

$$\sum_{k=0}^{n} \begin{Bmatrix} m+k \\ n \end{Bmatrix}_{V} \begin{Bmatrix} n \\ k \end{Bmatrix}_{U} (-1)^{\binom{k}{2}} \frac{1}{V_{k+rn+c}} = \Delta^{n} \frac{1}{V_{(r+1)n+c}} \left(\prod_{k=1}^{n} \frac{U_k}{V_k}\right)^{2} \begin{Bmatrix} (r+1)n+c-1 \\ n \end{Bmatrix}_{V}^{-1} \begin{Bmatrix} m-rn-c \\ n \end{Bmatrix}_{V}.$$

(3) For all positive odd n, positive m and all integers r and c such that $m-(r+1)n \geq c$ and $(r-1)n+c \geq 1$,

$$\sum_{k=0}^{n} \begin{Bmatrix} m+k \\ n \end{Bmatrix}_{U} \begin{Bmatrix} n \\ k \end{Bmatrix}_{U} (-1)^{\binom{k}{2}} U_{rn-k+c}^{-1} = (-1)^{\frac{n-1}{2}} \frac{1}{U_{(r-1)n+c}} \begin{Bmatrix} rn+m+c \\ n \end{Bmatrix}_{U} \begin{Bmatrix} rn+c \\ n \end{Bmatrix}_{U}^{-1},$$

$$\sum_{k=0}^{n} \begin{Bmatrix} m+k \\ n \end{Bmatrix}_{U} \begin{Bmatrix} n \\ k \end{Bmatrix}_{U} (-1)^{\binom{k}{2}} V_{rn-k+c}^{-1} = (-1)^{\frac{n-1}{2}} \frac{1}{V_{(r-1)n+c}} \begin{Bmatrix} rn+m+c \\ n \end{Bmatrix}_{V} \begin{Bmatrix} rn+c \\ n \end{Bmatrix}_{V}^{-1}.$$

(4) For positive odd n, positive m and all integers r and c such that $m-(r+1)n \geq c$ and $(r-1)n+c \geq 1$,

$$\sum_{k=0}^{n} \begin{Bmatrix} m+k \\ n \end{Bmatrix}_{V} \begin{Bmatrix} n \\ k \end{Bmatrix}_{U} (-1)^{\binom{k}{2}} \frac{1}{U_{rn-k+c}} = (-1)^{(n-1)/2} \frac{1}{U_{(r-1)n+c}} \begin{Bmatrix} rn+c \\ n \end{Bmatrix}_{U}^{-1} \begin{Bmatrix} m+rn+c \\ n \end{Bmatrix}_{V},$$

$$\sum_{k=0}^{n} \begin{Bmatrix} m+k \\ n \end{Bmatrix}_{V} \begin{Bmatrix} n \\ k \end{Bmatrix}_{U} (-1)^{\binom{k}{2}} \frac{1}{V_{rn-k+c}} = (-1)^{\frac{n-1}{2}} \Delta^{n} \left(\prod_{k=1}^{n} \frac{U_k}{V_k}\right)^{2} \begin{Bmatrix} rn+m+c \\ n \end{Bmatrix}_{U} \begin{Bmatrix} rn+c \\ n \end{Bmatrix}_{V}^{-1}.$$

(5) For positive n, m and all integers r and c such that $m-(r+1)\,n+c\geq 1$ and $rn+c\geq 1$,

$$\sum_{k=0}^{n}\begin{Bmatrix} m-k\\ n\end{Bmatrix}_U\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}U_{k+rn+c}^{-1}=\frac{1}{U_{(r+1)n+c}}\begin{Bmatrix} m+rn+c\\ n\end{Bmatrix}_U\begin{Bmatrix} (r+1)\,n+c-1\\ n\end{Bmatrix}_U^{-1},$$

$$\sum_{k=0}^{n}\begin{Bmatrix} m-k\\ n\end{Bmatrix}_U\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}V_{k+rn+c}^{-1}=\frac{1}{V_{(r+1)n+c}}\begin{Bmatrix} m+rn+c\\ n\end{Bmatrix}_V\begin{Bmatrix} (r+1)\,n+c-1\\ n\end{Bmatrix}_V^{-1}.$$

(6) For positive n, m and all integers r and c such that $m-(r+1)\,n+c\geq 1$ and $rn+c\geq 1$,

$$\sum_{k=0}^{n}\begin{Bmatrix} m-k\\ n\end{Bmatrix}_U\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}\frac{1}{U_{k+rn+c}}=\frac{1}{U_{rn+c}}\begin{Bmatrix} (r+1)\,n+c\\ n\end{Bmatrix}_U^{-1}\begin{Bmatrix} m+rn+c\\ n\end{Bmatrix}_V,$$

$$\sum_{k=0}^{n}\begin{Bmatrix} m-k\\ n\end{Bmatrix}_V\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}\frac{1}{V_{k+rn+c}}=\Delta^n\left(\prod_{k=1}^{n}\frac{U_k}{V_k}\right)^2\begin{Bmatrix} m+rn+c\\ n\end{Bmatrix}_U\begin{Bmatrix} (r+1)\,n+c\\ n\end{Bmatrix}_V^{-1}.$$

(7) For odd n, positive m, and, all integers r and c

$$\sum_{k=0}^{n}\begin{Bmatrix} m+k\\ n\end{Bmatrix}_U\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}U_{k+rn+c}=(-1)^{\frac{n-1}{2}}U_{n(m+r+1)+c},$$

$$\sum_{k=0}^{n}\begin{Bmatrix} m+k\\ n\end{Bmatrix}_U\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}V_{k+rn+c}=(-1)^{\frac{n-1}{2}}V_{n(m+r+1)+c}$$

and

$$\sum_{k=0}^{n}\begin{Bmatrix} m+k\\ n\end{Bmatrix}_V\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}U_{k+rn+c}=(-1)^{\frac{n-1}{2}}\Delta^{\frac{n-1}{2}}V_{n(m+r+1)+c}\prod_{k=1}^{n}\frac{U_k}{V_k},$$

$$\sum_{k=0}^{n}\begin{Bmatrix} m+k\\ n\end{Bmatrix}_V\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}V_{k+rn+c}=(-1)^{\frac{n-1}{2}}\Delta^{\frac{n+1}{2}}U_{n(m+r+1)+c}\prod_{k=1}^{n}\frac{U_k}{V_k}.$$

(8) For nonnegative integers n, m, and all integers r and c such that $m\geq n$

$$\sum_{k=0}^{n}\begin{Bmatrix} m-k\\ n\end{Bmatrix}_U\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}V_{rn-k+c}=V_{(m-n+r)n+c}$$

$$\sum_{k=0}^{n}\begin{Bmatrix} m-k\\ n\end{Bmatrix}_U\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}U_{rn-k+c}=U_{(m-n+r)n+c}$$

(9) For nonnegative integers n, m, and all integers r and c such that $m\geq n$

$$\sum_{k=0}^{n}\begin{Bmatrix} m-k\\ n\end{Bmatrix}_V\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}U_{rn-k+c}=\Delta^{\lfloor\frac{n}{2}\rfloor}\left(\prod_{k=1}^{n}\frac{U_k}{V_k}\right)\begin{cases} U_{(m-n+r)n+c} & \text{if } n \text{ is even,}\\ V_{(m-n+r)n+c} & \text{if } n \text{ is odd,}\end{cases}$$

and

$$\sum_{k=0}^{n}\begin{Bmatrix} m-k\\ n\end{Bmatrix}_U\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}V_{rn-k+c}=V_{(m-n+r)n+c},$$

$$\sum_{k=0}^{n}\begin{Bmatrix} m-k\\ n\end{Bmatrix}_U\begin{Bmatrix} n\\ k\end{Bmatrix}_U(-1)^{\binom{k}{2}}U_{rn-k+c}=U_{(m-n+r)n+c}.$$

As a showcase, we will show how the above generalized Fibonomial-Lucanomial-Fibonacci or Lucas corollaries are obtained from our main results. For this, we will prove the first corollary, that is,

$$\sum_{k=0}^{n} \left\{ \begin{matrix} m+k \\ n \end{matrix} \right\}_U \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U (-1)^{\binom{k}{2}} \frac{1}{U_{k+rn+c}} = \frac{1}{U_{(r+1)n+c}} \left\{ \begin{matrix} m-rn-c \\ n \end{matrix} \right\}_U \left\{ \begin{matrix} (r+1)\,n+c-1 \\ n \end{matrix} \right\}_U^{-1}$$

for positive even n. First we convert it into q-form by using the relationship given in the Introduction. Thus, after rearranging and simplifications, the claim takes the form

$$\begin{aligned} &\sum_{k=0}^{n} \alpha^{n(m+k-n)} \begin{bmatrix} m+k \\ n \end{bmatrix}_q \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{\binom{k}{2}} \frac{1-q}{\alpha^{k+rn+c-1}\left(1-q^{k+rn+c}\right)} \\ &= \frac{1-q}{\alpha^{(r+1)n+c-1}\left(1-q^{(r+1)n+c}\right)} \\ &\times \alpha^{n(m-rn-c-n)} \begin{bmatrix} m-rn-c \\ n \end{bmatrix}_q \alpha^{-n((r+1)n+c-1-n)} \begin{bmatrix} (r+1)\,n+c-1 \\ n \end{bmatrix}_q^{-1} \end{aligned}$$

or

$$\begin{aligned} &\sum_{k=0}^{n} \begin{bmatrix} m+k \\ n \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{\binom{k}{2}} \frac{1}{\left(1-q^{k+rn+c}\right)} \alpha^{-k+2kn-k^2} \\ &= \alpha^{-2n(c+nr)} \frac{1}{\left(1-q^{(r+1)n+c}\right)} \begin{bmatrix} m-rn-c \\ n \end{bmatrix}_q \begin{bmatrix} (r+1)\,n+c-1 \\ n \end{bmatrix}_q^{-1}. \end{aligned}$$

For the case of even n, we have to prove that

$$\begin{aligned} &\sum_{k=0}^{n} \begin{bmatrix} m+k \\ n \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{\left(1-q^{k+rn+c}\right)} (-1)^k q^{\binom{k+1}{2}-kn} \\ &= q^{n(c+nr)} \frac{1}{\left(1-q^{(r+1)n+c}\right)} \begin{bmatrix} m-rn-c \\ n \end{bmatrix}_q \begin{bmatrix} (r+1)\,n+c-1 \\ n \end{bmatrix}_q^{-1}. \end{aligned}$$

If we take $a = q^{rn+c}$ and $p = q$ in Theorem 1, we write

$$\sum_{k=0}^{n} \begin{bmatrix} m+k \\ n \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}-kn} \frac{1}{1-q^{k+rn+c}} = q^{n(rn+c)} \frac{(q^{-rn-c}qq^{m-n};q)_n}{(q^{rn+c};q)_{n+1}},$$

which, after some rearrangement, is equal to

$$\begin{aligned} &q^{n(rn+c)} \frac{\left(q^{1-c+m-(r+1)n};q\right)_n}{(q^{rn+c};q)_{n+1}} \\ &= q^{n(rn+c)} \frac{(q;q)_{m-rn-c}}{(q;q)_{n(r+1)+c}} \frac{(q;q)_{rn+c-1}}{(q;q)_{m-(r+1)n-c}} \\ &= q^{n(rn+c)} \frac{1}{1-q^{n(r+1)+c}} \frac{(q;q)_{m-rn-c}}{(q;q)_{m-(r+1)n-c}\,(q;q)_n} \frac{(q;q)_{rn+c-1}\,(q;q)_n}{(q;q)_{n(r+1)+c-1}} \\ &= q^{n(c+nr)} \frac{1}{\left(1-q^{(r+1)n+c}\right)} \begin{bmatrix} m-rn-c \\ n \end{bmatrix}_q \begin{bmatrix} (r+1)\,n+c-1 \\ n \end{bmatrix}_q^{-1}, \end{aligned}$$

as expected. So the claim is true.

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