SUMMABILITY ON MELLIN-TYPE NONLINEAR INTEGRAL OPERATORS

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Abstract. In this study, approximation properties of the Mellin-type nonlinear integral operators defined on multivariate functions are investigated. In order to get more general results than the classical aspects, we mainly use the summability methods defined by Bell. Considering the Haar measure with variation semi-norm in Tonelli's sense, we approach to the functions of bounded variation. Similar results are also obtained for uniformly continuous and bounded functions. Using suitable function classes we investigate the rate of convergence in the approximation. Finally, we give a non-trivial application verifying our approach.

1. Introduction

In this paper, we study the approximation properties of the Mellin-type nonlinear integral transforms which have some important applications in many areas, such as, optical physics, engineering, statistics, economics, signal process (see [10, 11, 15, 16, 17, 19, 25, 26]). We have mainly motivated from the recent papers by Angeloni and Vinti (see [5, 6]). We use the Haar measure, which is invariant under multiplicative group operation, instead of the Lebesgue measure. In the approximation to multivariate functions defined on the N-dimensional open interval $(0,\infty)^N$, we consider a general summability process rather than the usual convergence, which enables us to get more general results than the classical aspects.

It is known that a summability method is a common and useful method to handle the lack of the usual convergence. It is also used for acceleration of convergence rate of a sequence. We adopt the Bell-type summability methods which are more general than the Cesàro convergence [18] and almost convergence [22].

Recall the following definitions from the summability theory:

Let $\mathcal{A}=\{[a_{nk}^{\nu}]\}$ $(n,k,\nu\in\mathbb{N})$ be a family of infinite matrices. For a sequence $x=(x_k)$, $\mathcal{A}-$transform of x is a sequence $\mathcal{A}x:=\{(\mathcal{A}x)_n^{\nu}\}$ defined by $(\mathcal{A}x)_n^{\nu}=\sum_{k=1}^{\infty}a_{nk}^{\nu}x_k$ $(n,\nu\in\mathbb{N})$ if the series is convergent for every n,ν. Then x is $\mathcal{A}$-summable to a number L if $\lim_{n\to\infty}(\mathcal{A}x)_n^{\nu}=L$ uniformly in $\nu\in\mathbb{N}$. We will denote this convergence by

$$\mathcal{A}-\lim x=L.$$

In this method, regularity of a family of matrices play an important role since the usual approximation result becomes a special case of a regular summability process. For general properties of $\mathcal{A}$-summability methods, we refer to the papers [13, 14, 20, 21, 22].

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We say that a method $\mathcal{A}$ is regular if $\lim x = L$ implies $\mathcal{A}-\lim x = L$. A characterization for regularity of a method was given by Bell in [14]: "$\mathcal{A}$ is regular if and only if (a) for every $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{nk}^{\nu} = 0$ uniformly in ν; (b) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}^{\nu} = 1$ uniformly in ν; (c) for each $n, \nu \in \mathbb{N}$, $\sum_{k=1}^{\infty} |a_{nk}^{\nu}| < \infty$, and there exist integers N, M such that $\sup_{n\geq N, \nu\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}^{\nu}| \leq M$". Throughout the paper we will assume that the method $\mathcal{A}$ is regular together with nonnegative real entries.

In this paper, we investigate the Mellin-type nonlinear integral operators defined by

$$\mathcal{T}_{n,\nu}(f;\mathbf{s}) = \sum_{k=1}^{\infty} a_{nk}^{\nu} \int_{\mathbb{R}_+^N} K_k(\mathbf{t}, f(\mathbf{st})) \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \tag{1.1}$$

where $f \in L_{\mu}^{\infty}\left(\mathbb{R}_+^N\right)$ i.e., f is essentially bounded with respect to the Haar measure. Here we use the following notations:

- $\mathbb{R}_+^N = \{(x_1, x_2, \ldots, x_N) : x_i > 0 \text{ for } i = 1, 2, \ldots, N\}$,
- $\mathbf{s} = (s_1, s_2, \ldots, s_N)$, $\mathbf{t} = (t_1, t_2, \ldots, t_N) \in \mathbb{R}_+^N$,
- $\mathbf{st} = (s_1t_1, s_2t_2, \ldots, s_Nt_N)$,
- $\langle \mathbf{t} \rangle = t_1t_2 \ldots t_N$.

In (1.1), we adopt the followings:

- $\mathcal{A} = \{[a_{nk}^{\nu}]\}_{n=1}^{\infty}$ is a regular summability method,
- $K_k(\mathbf{s}, t) : \mathbb{R}_+^N \times \mathbb{R} \to \mathbb{R}$ is a family of kernels such that $K_k(\mathbf{s}, t) = L_k(\mathbf{s})H_k(t)$ for every $\mathbf{s} \in \mathbb{R}_+^N$, $t \in \mathbb{R}$ and $k \in \mathbb{N}$,
- $L_k : \mathbb{R}_+^N \to \mathbb{R}$ is a sequence of functions such that $L_k \in L_{\mu}^1\left(\mathbb{R}_+^N\right)$,
- $H_k : \mathbb{R} \to \mathbb{R}$ is a sequence of functions such that $H_k(0) = 0$ and H_k is Lipschitz, uniformly in $k \in \mathbb{N}$ i.e., there exists a constant $C > 0$ such that $|H_k(x) - H_k(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$.

Then, we observe that $\mathcal{T}_{n,\nu}(f;\mathbf{s})$ is well defined (see Proposition 2.1 for details). Our aim is to apply summability process on the nonlinear integral operators in (1.1). We should note that the usage of some summation techniques in the approximation by linear operators may be found in the papers [8, 9, 24, 27]. In this study, we generalize the results obtained by Angeloni and Vinti [5, 6]. More precisely, under the suitable conditions on L_k, we will get an approximation with respect to the variation semi-norm to an absolutely continuous function f of several variables by means of $\mathcal{T}_{n,\nu}(f)$. In this approximation we will use the Tonelli variation, which is more appropriate than the other definitions of bounded variation in N-dimension. Then we also evaluate the rate of convergence for suitable Lipschitz classes. The same process will be done for the classical uniform norm. Finally in the last section, we give a specific example verifying our approach.

2. Summability Process on the Operators (1.1) with respect to the Variation Semi-norm

In this section, we investigate the convergence in variation of the nonlinear operators given in (1.1).

We first need the following notations, which were considered in [1, 2, 3, 4, 5, 6, 7, 28, 29].

- $I = \prod_{i=1}^{N} [a_i, b_i]$ denotes an N-dimensional interval in $\mathbb{R}_+^N$.

- $I'_j := \left[a'_j, b'_j\right]$ denotes the $N-1$ dimensional interval, which is obtained by deleting the jth coordinate of I, i.e., $I'_j = \prod_{i=1,\ i\neq j}^{N} [a_i, b_i]$, so that $I = I'_j \times [a_j, b_j]$.
- For any vector $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N_+$ we write $\mathbf{x} = \left(x'_j, x_j\right)$ where $x'_j := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots x_N)$. Similarly, we also write $f\left(x'_j, x_j\right) = f(\mathbf{x})$ for any $f : \mathbb{R}^N_+ \to \mathbb{R}$.
- $L^1_\mu\left(\mathbb{R}^N_+\right)$ denotes the space of all functions $f : \mathbb{R}^N_+ \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^N_+} |f(\mathbf{t})| \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} < \infty,$$

where $\langle \mathbf{t} \rangle = t_1 t_2 \ldots t_N$.
- Let $V_{[a_j, b_j]}\left[f\left(x'_j, \cdot\right)\right]$ be the one dimensional Jordan variation (in the usual sense) of the jth section of f.
- Define the $(N-1)$-dimensional integral Φ_j by

$$\Phi_j(f, I) := \int_{a'_j}^{b'_j} V_{[a_j, b_j]}\left[f\left(x'_j, \cdot\right)\right] \frac{dx'_j}{\left\langle x'_j \right\rangle},$$

where $\left\langle x'_j \right\rangle = \prod_{i=1,\ i\neq j}^{N} x_i$.
- Φ denotes the Euclidean norm of the vector $(\Phi_1, \ldots, \Phi_N)$, that is

$$\Phi(f, I) := \left\{\sum_{k=1}^{N} \Phi^2_j(f, I)\right\}^{\frac{1}{2}},$$

where $\Phi(f, I) = \infty$ if $\Phi_j(f, I) = \infty$ for some $j = 1, 2, \ldots, N$.

Then, the variation semi-norm of f on $I \subset \mathbb{R}^N_+$ is defined as

$$V_I[f] := \sup \sum_{i=1}^{m} \Phi(f, J_i),$$

where the supremum is taken over all the finite families of N-dimensional intervals $\{J_1, J_2, \ldots, J_m\}$ which form partition of I. Then if we pass supremum over all the intervals $I \subset \mathbb{R}^N_+$, we get the variation of f on $\mathbb{R}^N_+$, namely

$$V[f] := \sup_{I \subset \mathbb{R}^N_+} V_I[f].$$

Now let

$$BV\left(\mathbb{R}^N_+\right) := \left\{f \in L^1_\mu\left(\mathbb{R}^N_+\right) : V[f] < \infty\right\}.$$

Notice that if $f \in BV\left(\mathbb{R}^N_+\right)$, then $f\left(x'_j, \cdot\right)$ is of bounded variation in the classical Jordan sense on $\mathbb{R}_+$ and $V_{\mathbb{R}_+}\left[f\left(x'_j, \cdot\right)\right] \in L^1_\mu\left(\mathbb{R}^{N-1}_+\right)$ for almost every $x'_j \in \mathbb{R}^{N-1}_+$.

A function $f : \mathbb{R}^N_+ \to \mathbb{R}$ is said to be locally absolutely continuous on $I = \prod_{i=1}^{N} [a_i, b_i] \subset \mathbb{R}^N_+$ if, for every $j = 1, \ldots, N$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for almost every $x'_j \in \mathbb{R}^{N-1}_+$ and for all collection of nonoverlapping intervals $[\alpha^\nu, \beta^\nu] \subset [a_j, b_j]$ $\nu = 1, \ldots, n$, $\sum_{\nu=1}^{n} (\beta^\nu - \alpha^\nu) < \delta$ implies $\sum_{\nu=1}^{n} \left|f\left(x'_j, \beta^\nu\right) - f\left(x'_j, \alpha^\nu\right)\right| < \varepsilon$. By $AC_{loc}\left(\mathbb{R}^N_+\right)$, we denote the set of all locally absolutely continuous functions. Finally, we define

$$AC\left(\mathbb{R}^N_+\right) := BV\left(\mathbb{R}^N_+\right) \cap AC_{loc}\left(\mathbb{R}^N_+\right),$$

which is a closed subspace of $BV\left(\mathbb{R}_{+}^{N}\right)$ with respect to the variation functional (see [5]).

We consider the following assumptions:

(i) There exists a constant $A>0$ such that $\sup\limits_{n,\nu\in\mathbb{N}}\sum_{k=1}^{\infty}a_{nk}^{\nu}\left\Vert L_{k}\right\Vert_{L_{\mu}^{1}}=A<\infty,$

(ii) $\mathcal{A}-\lim\left(\int\limits_{\mathbb{R}_{+}^{N}}L_{k}\left(\mathbf{t}\right)\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}\right)=1,$

(iii) for any $\delta>0$, $\mathcal{A}-\lim\left(\int\limits_{\left\vert\mathbf{1}-\mathbf{t}\right\vert\geq\delta}L_{k}\left(\mathbf{t}\right)\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}\right)=0$, where $\mathbf{1}:=(1,1,\cdots,1)$,

(iv) $\frac{V_{J}\left[G_{k}\right]}{m(J)}\rightarrow0$ as $k\rightarrow\infty$ uniformly in every proper bounded interval $J\subset\mathbb{R}$, where $G_{k}\left(u\right):=H_{k}\left(u\right)-u$; $V_{J}\left[G_{k}\right]$ is the usual one dimensional Jordan variation of G_{k} and $m\left(J\right)$ is the length of the interval $J\subset\mathbb{R}$.

Remark 2.1. *Due to the nonlinearity of the kernels of our integral operators, we need to use assumption* (iv). *Notice that condition* (iv) *implies the Lipschitz property of* H_{k}*, asymptotically* (*see, for details,* [6]). *But then, the sum in the definition of* $\mathcal{T}_{n,\nu}$ *would start from a sufficiently large number* k_{0}. *On the other hand, our operators* (1.1) *may be written as follows:*

$$\mathcal{T}_{n,\nu}\left(f;\mathbf{s}\right)=\sum_{k=1}^{\infty}a_{nk}^{\nu}T_{k}(f;\mathbf{s}),$$

where

$$T_{k}\left(f;\mathbf{s}\right):=\int_{\mathbb{R}_{+}^{N}}K_{k}(\mathbf{t},f(\mathbf{st}))\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}. \tag{2.1}$$

Hence, if one takes $\mathcal{A}=\{\mathcal{I}\}$*, the identity matrix, then we immediately get the operators (2.1) which were considered in* [5, 6]. *In this case, our conditions* $(i)-(iii)$ *reduce to the ones in* [5, 6].

Next result gives that the operator $\mathcal{T}_{n,\nu}$ is well-defined.

Proposition 2.1. *Let* $\mathcal{A}=\{[a_{nk}^{\nu}]\}$ *be a nonnegative regular summability method. Then for all* $f\in L_{\mu}^{\infty}\left(\mathbb{R}_{+}^{N}\right)$, $\mathcal{T}_{n,\nu}(f;\mathbf{s})<\infty$ *for all* $s\in\mathbb{R}_{+}^{N}$. *Moreover, if the condition* (i) *holds, then* $\mathcal{T}_{n,\nu}(f)\in L_{\mu}^{1}\left(\mathbb{R}_{+}^{N}\right)$ *for every* $f\in L_{\mu}^{1}\left(\mathbb{R}_{+}^{N}\right)$.

Proof. By the definition of the operator (1.1), using Hölder inequality it can be seen that

$$\begin{aligned}\left\vert\mathcal{T}_{n,\nu}(f;\mathbf{s})\right\vert&\leq\sum_{k=1}^{\infty}a_{nk}^{\nu}\int\limits_{\mathbb{R}_{+}^{N}}\left\vert L_{k}(\mathbf{t})\right\vert\left\vert H_{k}\left(f(\mathbf{st})\right)\right\vert\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}\\&\leq C\sum_{k=1}^{\infty}a_{nk}^{\nu}\int\limits_{\mathbb{R}_{+}^{N}}\left\vert L_{k}(\mathbf{t})\right\vert\left\vert f(\mathbf{st})\right\vert\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}\leq C\sum_{k=k_{0}}^{\infty}a_{nk}^{\nu}\left\Vert L_{k}\right\Vert_{L_{\mu}^{1}}\left\Vert f\right\Vert_{L_{\mu}^{\infty}}\\&\leq CA\left\Vert f\right\Vert_{L_{\mu}^{\infty}}\end{aligned}$$

for every $\mathbf{s} \in \mathbb{R}_+^N$. Moreover considering the condition (i), by the Fubini-Tonelli theorem we obtain that

$$\int\limits_{\mathbb{R}_+^N} |\mathcal{T}_{n,\nu}(f;\mathbf{s})| \frac{\mathbf{ds}}{\langle \mathbf{s} \rangle} \leq \int\limits_{\mathbb{R}_+^N} \left(\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} |L_k(\mathbf{t})| \, |H_k(f(\mathbf{st}))| \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \right) \frac{\mathbf{ds}}{\langle \mathbf{s} \rangle}$$

$$\leq C \int\limits_{\mathbb{R}_+^N} \sum_{k=1}^{\infty} a_{nk}^{\nu} |L_k(\mathbf{t})| \left(\int\limits_{\mathbb{R}_+^N} |f(\mathbf{st})| \frac{\mathbf{ds}}{\langle \mathbf{s} \rangle} \right) \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle}$$

$$\leq C \sum_{k=1}^{\infty} a_{nk}^{\nu} \|L_k\|_{L_\mu^1} \|f\|_{L_\mu^1} \leq CA \|f\|_{L_\mu^1},$$

which gives the result. □

We also get the following result.

Proposition 2.2. *Let $\mathcal{A} = \{[a_{nk}^{\nu}]\}$ be a nonnegative regular summability method. If $f \in BV(\mathbb{R}_+^N)$ and condition (i) holds, then we have*

$$V[\mathcal{T}_{n,\nu}(f)] \leq (CA) V[f].$$

Proof. Let $I = \Pi_{i=1}^N [a_i, b_i] \subset \mathbb{R}_+^N$ and let $\{J_1, \ldots, J_m\}$ be a partition of I with $J_q = \Pi_{j=1}^N [{}^q a_j, {}^q b_j]$, $q = 1, 2, \ldots m$. For every fixed $j = 1, \ldots, N$ and $q = 1, \ldots, m$ assume that $\left\{ s_j^0 = {}^q a_j, \ldots, s_j^\lambda = {}^q b_j \right\}$ is a partition of the interval $[{}^q a_j, {}^q b_j]$. Then for each $s_j' \in I_j'$, we obtain that

$$S_j := \sum_{\mu=1}^{\lambda} \left| \mathcal{T}_{n,\nu} f\left(s_j', s_j^\mu\right) - \mathcal{T}_{n,\nu} f\left(s_j', s_j^{\mu-1}\right) \right|$$

$$\leq \sum_{\mu=1}^{\lambda} \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} |L_k(\mathbf{t})| \left| H_k\left(\tau_{\mathbf{t}} f\left(s_j', s_j^\mu\right)\right) - H_k\left(\tau_{\mathbf{t}} f\left(s_j', s_j^{\mu-1}\right)\right) \right| \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle}$$

$$\leq C \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} |L_k(\mathbf{t})| \sum_{\mu=1}^{\lambda} \left| \tau_{\mathbf{t}} f\left(s_j', s_j^\mu\right) - \tau_{\mathbf{t}} f\left(s_j', s_j^{\mu-1}\right) \right| \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle}$$

$$\leq C \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} |L_k(\mathbf{t})| V_{[{}^q a_j, {}^q b_j]} \left[\tau_{\mathbf{t}} f\left(s_j', \cdot\right) \right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle},$$

where $\tau_{\mathbf{t}} f(\mathbf{s}) := f(\mathbf{st})$ $\mathbf{s}, \mathbf{t} \in \mathbb{R}_+^N$ denotes the dilation operator. Then passing to the supremum over all partitions of $[{}^q a_j, {}^q b_j]$, we get

$$V_{[{}^q a_j, {}^q b_j]} \left[\mathcal{T}_{n,\nu} f\left(s_j', \cdot\right) \right] \leq C \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} |L_k(\mathbf{t})| V_{[{}^q a_j, {}^q b_j]} \left[\tau_{\mathbf{t}} f\left(s_j', \cdot\right) \right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle}.$$

By the Fubini-Tonelli theorem, one can see that

$$\begin{aligned}\Phi_j\left(\mathcal{T}_{n,\nu}(f), J_q\right) &\leq C \int\limits_{a_j'}^{b_j'} \left(\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| V_{[{}^q a_j, {}^q b_j]}\left[\tau_{\mathbf{t}} f\left(s_j', \cdot\right)\right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \right) \frac{ds_j'}{\langle s_j' \rangle} \\ &= C \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \left(\int\limits_{a_j'}^{b_j'} V_{[{}^q a_j, {}^q b_j]}\left[\tau_{\mathbf{t}} f\left(s_j', \cdot\right)\right] \frac{ds_j'}{\langle s_j' \rangle} \right) \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\ &= C \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \Phi_j\left(\tau_{\mathbf{t}} f, J_q\right) \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle}.\end{aligned}$$

Using the generalized Minkowski-type inequality it is not hard to see that

$$\begin{aligned}\Phi\left(\mathcal{T}_{n,\nu}(f), J_q\right) &\leq C \left\{ \sum_{j=1}^{N} \left(\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \Phi_j\left(\tau_{\mathbf{t}}(f), J_q\right) \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \right)^2 \right\}^{\frac{1}{2}} \\ &\leq C \sum_{k=1}^{\infty} a_{nk}^{\nu} \left\{ \sum_{j=1}^{N} \left(\int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \Phi_j\left(\tau_{\mathbf{t}}(f), J_q\right) \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \right)^2 \right\}^{\frac{1}{2}} \\ &\leq C \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \left\{ \sum_{j=1}^{N} \left(\Phi_j\left(\tau_{\mathbf{t}}(f), J_q\right)\right)^2 \right\}^{\frac{1}{2}} \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\ &= C \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \Phi\left(\tau_{\mathbf{t}}(f), J_q\right) \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle}.\end{aligned}$$

Now summing over $q = 1, \ldots, m$ and passing to the supremum over the partitions of $\{J_1, \ldots, J_m\}$ then we get the following:

$$V_I\left[\mathcal{T}_{n,\nu}(f)\right] \leq C \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| V_I\left[\tau_{\mathbf{t}}(f)\right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle}.$$

Since $I \subset \mathbb{R}_+^N$ is an arbitrary interval, by (i)

$$\begin{aligned}V\left[\mathcal{T}_{n,\nu}(f)\right] &\leq C \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| V\left[\tau_{\mathbf{t}}(f)\right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\ &\leq CV[f] \sum_{k=1}^{\infty} a_{nk}^{\nu} \left\|L_k\right\|_{L_\mu^1} = (CA)V[f]\end{aligned}$$

holds for all $n, \nu \in \mathbb{N}$. $\square$

Now we get the first approximation result with respect to the variation seminorm.

Theorem 2.3. *Let $\mathcal{A} = \{[a_{nk}^{\nu}]\}$ be a nonnegative regular summability method. Assume that conditions $(i) - (iv)$ hold. Then, we have*

$$\lim_{n\to\infty} V\left[\mathcal{T}_{n,\nu}(f) - f\right] = 0, \text{ uniformly in } \nu \in \mathbb{N},$$

for every $f \in AC\left(\mathbb{R}_+^N\right)$.

Proof. Using the same notation in the proof of Proposition 2.2, we may write from the triangle inequality and condition (i) that

$$\begin{aligned}
S_j &:= \sum_{\mu=1}^{\lambda} \left| (\mathcal{T}_{n,\nu} f)\left(s_j', s_j^{\mu}\right) - f\left(s_j', s_j^{\mu}\right) - \left[(\mathcal{T}_{n,\nu} f)\left(s_j', s_j^{\mu-1}\right) - f\left(s_j', s_j^{\mu-1}\right)\right]\right| \\
&\leq \sum_{\mu=1}^{\lambda} \sum_{k=1}^{\infty} a_{nk}^{\nu} \int_{\mathbb{R}_+^N} |L_k(\mathbf{t})| \left| H_k\left(f\left(s_j' t_j', s_j^{\mu} t_j\right)\right) - f\left(s_j' t_j', s_j^{\mu} t_j\right) \right. \\
&\left. - \left[H_k\left(f\left(s_j' t_j', s_j^{\mu-1} t_j\right)\right) - f\left(s_j' t_j', s_j^{\mu-1} t_j\right)\right]\right| \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\
&+ \sum_{\mu=1}^{\lambda} \sum_{k=1}^{\infty} a_{nk}^{\nu} \int_{\mathbb{R}_+^N} |L_k(\mathbf{t})| \left| f\left(s_j' t_j', s_j^{\mu} t_j\right) - f\left(s_j', s_j^{\mu}\right) \right. \\
&\left. - \left[f\left(s_j' t_j', s_j^{\mu-1} t_j\right) - f\left(s_j', s_j^{\mu-1}\right)\right]\right| \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\
&+ \left| \sum_{k=1}^{\infty} a_{nk}^{\nu} \int_{\mathbb{R}_+^N} L_k(\mathbf{t}) \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} - 1 \right| \sum_{\mu=1}^{\lambda} \left| f\left(s_j', s_j^{\mu}\right) - f\left(s_j', s_j^{\mu-1}\right)\right|
\end{aligned}$$

holds. Now passing to the supremum over the partitions of the interval $[{}^q a_j, {}^q b_j]$ we get the next inequality

$$\begin{aligned}
& V_{[{}^q a_j, {}^q b_j]}\left[\mathcal{T}_{n,\nu} f\left(s_j', \cdot\right) - f\left(s_j', \cdot\right)\right] \\
&\leq \sum_{k=1}^{\infty} a_{nk}^{\nu} \int_{\mathbb{R}_+^N} |L_k(\mathbf{t})| \, V_{[{}^q a_j, {}^q b_j]}\left[\tau_{\mathbf{t}}\left(H_k \circ f - f\right)\left(s_j', \cdot\right)\right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\
&+ \sum_{k=1}^{\infty} a_{nk}^{\nu} \int_{\mathbb{R}_+^N} |L_k(\mathbf{t})| \, V_{[{}^q a_j, {}^q b_j]}\left[\left(\tau_{\mathbf{t}}\left(f\right) - f\right)\left(s_j', \cdot\right)\right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\
&+ V_{[{}^q a_j, {}^q b_j]}\left[f\left(s_j', \cdot\right)\right] \left| \sum_{k=1}^{\infty} a_{nk}^{\nu} \int_{\mathbb{R}_+^N} L_k(\mathbf{t}) \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} - 1 \right|.
\end{aligned}$$

Hence, the Fubini-Tonelli theorem implies that

$$\begin{aligned}
&\Phi_j\left(\mathcal{T}_{n,\nu}(f)-f,J_q\right)\\
&\leq \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \Phi_j\left(\tau_{\mathbf{t}}\left(H_k\circ f-f\right),J_q\right)\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}\\
&+\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \Phi_j\left(\tau_{\mathbf{t}}\left(f\right)-f,J_q\right)\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}\\
&+\Phi_j\left[f,J_q\right]\left|\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} L_k(\mathbf{t})\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}-1\right|.
\end{aligned}$$

Using the generalized Minkowski inequality, we obtain that

$$\begin{aligned}
&\Phi\left(\mathcal{T}_{n,\nu}(f)-f,J_q\right)\\
&\leq \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \Phi\left(\tau_{\mathbf{t}}\left(H_k\circ f-f\right),J_q\right)\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}\\
&+\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \Phi\left(\tau_{\mathbf{t}}\left(f\right)-f,J_q\right)\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}\\
&+\Phi\left(f,J_q\right)\left|\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} L_k(\mathbf{t})\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}-1\right|.
\end{aligned}$$

Summing over $q=1,\ldots,m$ and after passing to the supremum over all the partitions of N-dimensional interval I, one can see that

$$\begin{aligned}
&V_I\left[\mathcal{T}_{n,\nu}(f)-f\right]\\
&\leq \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| V_I\left[\tau_{\mathbf{t}}\left(H_k\circ f-f\right)\right]\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}\\
&+\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| V_I\left[\tau_{\mathbf{t}}\left(f\right)-f\right]\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}\\
&+V_I\left[f\right]\left|\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} L_k(\mathbf{t})\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}-1\right|
\end{aligned}$$

holds. Since $I \subset \mathbb{R}_+^N$ is arbitrary, we derive the following inequality:

$$\begin{aligned} V\left[\mathcal{T}_{n,\nu}(f)-f\right] &\leq \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| V\left[\tau_{\mathbf{t}}\left(H_k \circ f - f\right)\right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\ &+ \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| V\left[\tau_{\mathbf{t}}\left(f\right)-f\right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\ &+ V\left[f\right] \left| \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} L_k(\mathbf{t}) \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} - 1 \right| \\ &=: I_1(n,\nu) + I_2(n,\nu) + I_3(n,\nu). \end{aligned}$$

It follows from the properties of variation in $\mathbb{R}_+^N$ that

$$V\left[\tau_{\mathbf{t}}\left(H_k \circ f - f\right)\right] = V\left[H_k \circ f - f\right]$$

for every $\mathbf{t} \in \mathbb{R}_+^N$. Now, if we consider the hypothesis (iv), then Proposition 3.3 in [6] implies that, for a given $\epsilon > 0$ there exists k_0 such that $V\left[H_k \circ f - f\right] < \varepsilon$ for every $k > k_0$. Then, we can divide $I_1(n,\nu)$ into two parts as follows:

$$\begin{aligned} &\sum_{k=1}^{k_0} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| V\left[H_k \circ f - f\right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} + \sum_{k=k_0+1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| V\left[H_k \circ f - f\right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\ &=: I_1^1(n,\nu) + I_1^2(n,\nu). \end{aligned}$$

We observe that

$$I_1^1(n,\nu) \leq M \sum_{k=1}^{k_0} a_{nk}^{\nu},$$

where $M = \max_{1 \leq k \leq k_0} \left\{ V\left[H_k \circ f - f\right] \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \right\}$. We also get

$$I_1^2(n,\nu) \leq \varepsilon \sum_{k=k_0+1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle}.$$

Now using the regularity of $\mathcal{A}$ and also considering the hypothesis (i), both $I_1^1(n,\nu)$ and $I_1^2(n,\nu)$ converge to zero as n tends to infinity (uniformly in ν).

On the other hand, from Theorem 1 in [5], for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\left|\mathbf{1} - \mathbf{t}\right| < \delta$ implies

$$V\left[\tau_{\mathbf{t}}\left(f\right)-f\right] < \varepsilon. \tag{2.2}$$

So, we divide $I_2(n,\nu)$ into two parts as follows:

$$\begin{aligned} &\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\left|\mathbf{1}-\mathbf{t}\right| < \delta} \left|L_k(\mathbf{t})\right| V\left[\tau_{\mathbf{t}}\left(f\right)-f\right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} + \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\left|\mathbf{1}-\mathbf{t}\right| \geq \delta} \left|L_k(\mathbf{t})\right| V\left[\tau_{\mathbf{t}}\left(f\right)-f\right] \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\ &:= I_2^1(n,\nu) + I_2^2(n,\nu). \end{aligned}$$

Now considering (iii) and (2.2), it is obvious that

$$\begin{aligned} I_2^1(n,\nu)+I_2^2(n,\nu) &\leq A\varepsilon + V\left[\tau_{\mathbf{t}}\left(f\right)-f\right]\varepsilon \\ &\leq \left(A+2V\left[f\right]\right)\varepsilon \end{aligned}$$

holds for sufficiently large $n\in\mathbb{N}$ and for each $\nu\in\mathbb{N}$, which means that $I_2^1(n,\nu)$ and $I_2^2(n,\nu)$ converge to zero as n tends to infinity (uniformly in ν).

Finally, condition (ii) guarantees that $I_3(n,\nu)$ also goes to zero as n tends to infinity (uniformly in ν). Therefore, the proof is completed. □

Now we study the rate of approximation for the operators in (1.1) using the suitable Lipschitz class of functions of $AC\left(\mathbb{R}_+^N\right)$. To evaluate the order of approximation we need the following assumptions which are observed by modifying the assumptions $(ii)\,,(iii)$ and (iv).

Let $0<\alpha\leq 1$. Then we will assume that

$$\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} L_k\left(\mathbf{t}\right)\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}-1=O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \nu), \tag{2.3}$$

for any fixed $\delta>0$,

$$\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{|\mathbf{1}-\mathbf{t}|<\delta} \left|L_k\left(\mathbf{t}\right)\right|\left|\log\mathbf{t}\right|^{\alpha}\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}=O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \nu), \tag{2.4}$$

$$\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{|\mathbf{1}-\mathbf{t}|\geq\delta} \left|L_k\left(\mathbf{t}\right)\right|\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}=O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \nu). \tag{2.5}$$

We use the following class of functions, which was introduced in [5, 6]:

$$VLip^N\left(\alpha\right):=\left\{f\in AC\left(\mathbb{R}_+^N\right): V\left[\tau_{\mathbf{t}}f-f\right]=O\left(\left|\log\mathbf{t}\right|^{\alpha}\right),\ \text{as } \left|\mathbf{1}-\mathbf{t}\right|\to 0\right\},$$

where $\log\mathbf{t}:=\left(\log t_1,\ldots,\log t_N\right),\ \mathbf{t}\in\mathbb{R}_+^N$.

Theorem 2.4. *Let $\{L_k\}$ be a sequence of kernels such that $\sup_{k\in\mathbb{N}}\left\|L_k\right\|_{L_\mu^1}=M<\infty$ for some $M>0$, and let $\alpha\in(0,1]$. Assume that (2.3), (2.4), (2.5) hold. Assume further that $\{\beta_k\}$ is a null sequence of positive real numbers satisfying that*

$$\frac{V_J[G_k]}{m(J)}\leq\beta_k \quad \text{(for all } k\in\mathbb{N} \text{ and for every bounded interval } J\subset\mathbb{R}) \tag{2.6}$$

and

$$\sum_{k=1}^{\infty} a_{nk}^{\nu}\beta_k=O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \nu). \tag{2.7}$$

Then for every $f\in VLip^N\left(\alpha\right)$

$$V\left[\mathcal{T}_{n,\nu}f-f\right]=O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \nu).$$

Proof. By the proof of the Theorem 2.3 it is obvious that

$$\begin{aligned}
V\left[\mathcal{T}_{n,\nu}f-f\right] &\leq \sum_{k=1}^{\infty} a_{nk}^{\nu} V\left[H_k\circ f-f\right]\int\limits_{\mathbb{R}_+^N}\left|L_k(\mathbf{t})\right|\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle} \\
&+\sum_{k=1}^{\infty} a_{nk}^{\nu}\int\limits_{\mathbb{R}_+^N}\left|L_k(\mathbf{t})\right| V\left[\tau_{\mathbf{t}}\left(f\right)-f\right]\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle} \\
&+V\left[f\right]\left|\sum_{k=1}^{\infty} a_{nk}^{\nu}\int\limits_{\mathbb{R}_+^N} L_k(\mathbf{t})\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}-1\right| \\
&:= J_1(n,\nu)+J_2(n,\nu)+J_3(n,\nu).
\end{aligned}$$

About $J_1(n,\nu)$, by Proposition 4.1 in [6], (2.6) implies that $V\left[H_k\circ f-f\right]\leq\beta_k$ and using the hypothesis $\sup_{k\in\mathbb{N}}\left\|L_k\right\|_{L_\mu^1}=M<\infty$ it follows that

$$J_1(n,\nu)\leq M\sum_{k=1}^{\infty} a_{nk}^{\nu}\beta_k,$$

which implies

$$J_1(n,\nu)=O(n^{-\alpha}) \text{ as } n\to\infty \text{ (uniformly in } \nu). \tag{2.8}$$

due to the assumption (2.7). About $J_2(n,\nu)$, since $f\in V Lip^N\left(\alpha\right)$, there exists K, $\delta>0$ such that $V\left[\tau_{\mathbf{t}}f-f\right]\leq K\left|\log\mathbf{t}\right|^{\alpha}$ whenever $\left|\mathbf{1}-\mathbf{t}\right|<\delta$. Then it is not hard to see that

$$\begin{aligned}
J_2(n,\nu) &= \sum_{k=1}^{\infty} a_{nk}^{\nu}\int\limits_{\left|\mathbf{1}-\mathbf{t}\right|<\delta}\left|L_k(\mathbf{t})\right| V\left[\tau_{\mathbf{t}}\left(f\right)-f\right]\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle} \\
&+\sum_{k=1}^{\infty} a_{nk}^{\nu}\int\limits_{\left|\mathbf{1}-\mathbf{t}\right|\geq\delta}\left|L_k(\mathbf{t})\right| V\left[\tau_{\mathbf{t}}\left(f\right)-f\right]\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle} \\
&\leq K\sum_{k=1}^{\infty} a_{nk}^{\nu}\int\limits_{\left|\mathbf{1}-\mathbf{t}\right|<\delta}\left|L_k(\mathbf{t})\right|\left|\log\mathbf{t}\right|^{\alpha}\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle} \\
&+2V\left[f\right]\sum_{k=1}^{\infty} a_{nk}^{\nu}\int\limits_{\left|\mathbf{1}-\mathbf{t}\right|\geq\delta}\left|L_k(\mathbf{t})\right|\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle}.
\end{aligned}$$

Then, using (2.4) and (2.5), we get

$$J_2(n,\nu)=O(n^{-\alpha}) \text{ as } n\to\infty \text{ (uniformly in } \nu). \tag{2.9}$$

Finally, from (2.3), we immediately see that

$$J_3(n,\nu)=O(n^{-\alpha}) \text{ as } n\to\infty \text{ (uniformly in } \nu). \tag{2.10}$$

So, combining (2.8), (2.9) and (2.10), the proof is completed. □

3. Approximation by the Operators (1.1) with respect to the Uniform Norm

In this section we study the approximation properties of the nonlinear operator (1.1) by using the classical uniform norm on $\mathbb{R}_{+}^{N}$, denoted by $\|\cdot\|_{\infty}$.

Let $|\mathbf{x}|$ be the Euclidean norm of the N-dimensional vector $\mathbf{x}$. We say that a real-valued function defined on $\mathbb{R}_{+}^{N}$ is log-uniformly continuous if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}_{+}^{N}$ satisfying $|\log \mathbf{x} - \log \mathbf{y}| < \delta$ (see, for instance, [12, 23]). By $B(\mathbb{R}_{+}^{N})$ and $UC_{\log}(\mathbb{R}_{+}^{N})$ we denote the spaces of all bounded functions and all log-uniformly continuous functions on $\mathbb{R}_{+}^{N}$, respectively. We also define $BUC_{\log}(\mathbb{R}_{+}^{N}) := UC_{\log}(\mathbb{R}_{+}^{N}) \cap B(\mathbb{R}_{+}^{N})$.

In the uniform approximation, we need the following assumption instead of (iv):

$(iv)'$ $\|G_k\|_J \to 0$ as $k \to \infty$ for every bounded interval $J \subset \mathbb{R}$, where $G_k(u) = H_k(u) - u$ as stated before and $\|\cdot\|_J$ denotes the usual uniform norm on the interval J.

We first get the next result.

Proposition 3.1. *Let $\mathcal{A} = \{[a_{nk}^{\nu}]\}$ be a nonnegative regular summability method. If $f \in B(\mathbb{R}_{+}^{N})$ and (i) holds, then there exists a positive constant D such that*

$$\|\mathcal{T}_{n,\nu}(f)\|_{\infty} \leq D\,\|f\|_{\infty}$$

for every $n, \nu \in \mathbb{N}$, which implies $\mathcal{T}_{n,\nu}\left(B(\mathbb{R}_{+}^{N})\right) \subset B(\mathbb{R}_{+}^{N})$.

Proof. By the definition of the operator (1.1), one can see that

$$\begin{aligned}
|\mathcal{T}_{n,\nu}(f;\mathbf{s})| &\leq \sum_{k=1}^{\infty} a_{nk}^{\nu} \int_{\mathbb{R}_{+}^{N}} |L_k(\mathbf{t})|\,|H_k(f(\mathbf{st}))|\,\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle} \\
&\leq C \sum_{k=1}^{\infty} a_{nk}^{\nu} \int_{\mathbb{R}_{+}^{N}} |L_k(\mathbf{t})|\,|f(\mathbf{st})|\,\frac{\mathbf{dt}}{\langle\mathbf{t}\rangle} \\
&\leq C\,\|f\|_{\infty} \sum_{k=k_0}^{\infty} a_{nk}^{\nu}\,\|L_k\|_{L_{\mu}^{1}} \\
&\leq CA\,\|f\|_{\infty},
\end{aligned}$$

which gives

$$\|\mathcal{T}_{n,\nu}(f)\|_{\infty} \leq D\,\|f\|_{\infty}$$

with $D := CA$. Here the constants C and A come from the Lipschitz property and condition (i). □

Then, we obtain the following approximation result with respect to the uniform norm.

Theorem 3.2. *Let $\mathcal{A} = \{[a_{nk}^{\nu}]\}$ be a nonnegative regular summability method. Assume that conditions $(i)-(iii)$ and $(iv)'$ hold. Then, for every $f \in BUC_{\log}(\mathbb{R}_{+}^{N})$, we have*

$$\lim_{n\to\infty} \|\mathcal{T}_{n,\nu}(f) - f\|_{\infty} \to 0, \ \textit{uniformly in } \nu \in \mathbb{N},$$

or, equivalently,

$$\mathcal{A} - \lim T_k(f) = f,$$

where $\mathcal{T}_{n,\nu}$ and T_k are given by (1.1) and (2.1), respectively.

Proof. Using the triangle inequality, we get the following

$$\begin{aligned}
\|\mathcal{T}_{n,\nu}(f)-f\|_{\infty} &\leq \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_{+}^{N}} |L_k(\mathbf{t})| \left\|\tau_{\mathbf{t}}\left(H_k\circ f-f\right)\right\|_{\infty} \frac{\mathbf{dt}}{\langle \mathbf{t}\rangle} \\
&+ \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_{+}^{N}} |L_k(\mathbf{t})| \left\|\tau_{\mathbf{t}}(f)-f\right\|_{\infty} \frac{\mathbf{dt}}{\langle \mathbf{t}\rangle} \\
&+ \|f\|_{\infty} \left| \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_{+}^{N}} L_k(\mathbf{t}) \frac{\mathbf{dt}}{\langle \mathbf{t}\rangle} - 1 \right| \\
&:= I_1(n,\nu)+I_2(n,\nu)+I_3(n,\nu).
\end{aligned}$$

Observe that $\left\|\tau_{\mathbf{t}}\left(H_k\circ f-f\right)\right\|_{\infty}=\left\|H_k\circ f-f\right\|_{\infty}=\left\|G_k\left(f\right)\right\|_{\infty}$ for every $\mathbf{t}\in\mathbb{R}_{+}^{N}$. On the other hand since f is bounded, one can conclude that

$$|G_k\left(f\left(\mathbf{x}\right)\right)| \leq \|G_k\|_J = \sup_{u\in J} |H_k\left(u\right)-u|,$$

where $J=[C_1,C_2]$ and C_1,C_2 are the minimum and maximum values of f on $\mathbb{R}_{+}^{N}$ respectively. Hence by $(iv)'$, for every $\varepsilon>0$ there exists a number k_0 such that $\|G_k\|_J<\varepsilon$ for all $k>k_0$. Then the sum $I_1(n,\nu)$ becomes

$$\begin{aligned}
I_1(n,\nu) &\leq \sum_{k=1}^{k_0} a_{nk}^{\nu} \|G_k\|_J \int\limits_{\mathbb{R}_{+}^{N}} |L_k(\mathbf{t})| \frac{\mathbf{dt}}{\langle \mathbf{t}\rangle} + \varepsilon A \\
&\leq D \sum_{k=1}^{k_0} a_{nk}^{\nu} + \varepsilon A,
\end{aligned}$$

where $D:=\max_{1\leq k\leq k_0}\left(\|G_k\|_J \int\limits_{\mathbb{R}_{+}^{N}} |L_k(\mathbf{t})| \frac{\mathbf{dt}}{\langle \mathbf{t}\rangle}\right)$. Also considering the regularity of $\mathcal{A}$, we immediately see that

$$\lim_{n\to\infty} I_1(n,\nu)=0, \quad \text{uniformly in } \nu.$$

Since f is log-uniformly continuous on $\mathbb{R}_{+}^{N}$, for every $\varepsilon>0$ there exists a $\gamma>0$ such that $|\log\left(\mathbf{st}\right)-\log\left(\mathbf{s}\right)|<\gamma$ implies $|f\left(\mathbf{st}\right)-f\left(\mathbf{s}\right)|<\varepsilon$. Using the fact that

$$|\log\left(\mathbf{st}\right)-\log\left(\mathbf{s}\right)| = |\log\left(\mathbf{s}\right)+\log\left(\mathbf{t}\right)-\log\left(\mathbf{s}\right)| = |\log\left(\mathbf{t}\right)|,$$

we observe that $|\log\mathbf{t}|<\gamma$ implies $|f\left(\mathbf{st}\right)-f\left(\mathbf{s}\right)|<\varepsilon$. Also since $|\log\mathbf{t}|\to 0$ as $|\mathbf{1}-\mathbf{t}|\to 0$, for a given $\gamma>0$, there is a $\delta>0$ such that $|\log\mathbf{t}|<\gamma$ whenever

$|\mathbf{1}-\mathbf{t}|<\delta$. Hence, taking into account the condition (i), we get

$$\begin{aligned} I_2(n,\nu) &= \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{|\mathbf{1}-\mathbf{t}|<\delta} |L_k(\mathbf{t})| \, \|\tau_{\mathbf{t}} f - f\|_{\infty} \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\ &\quad + \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{|\mathbf{1}-\mathbf{t}|\geq\delta} |L_k(\mathbf{t})| \, \|\tau_{\mathbf{t}} f - f\|_{\infty} \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle} \\ &< A\varepsilon + 2\|f\|_{\infty} \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{|\mathbf{1}-\mathbf{t}|\geq\delta} |L_k(\mathbf{t})| \frac{\mathbf{dt}}{\langle \mathbf{t} \rangle}. \end{aligned}$$

Hence, from the assumption (iii) we obtain that

$$\lim_{n\to\infty} I_2(n,\nu)=0, \quad \text{uniformly in } \nu.$$

Finally, condition (ii) immediately gives that

$$\lim_{n\to\infty} I_3(n,\nu)=0, \quad \text{uniformly in } \nu.$$

Combining the above results, the proof is completed. □

Now we investigate the order of approximation. For this, in addition to the conditions (2.3), (2.4) and (2.5) we need the following Lipschitz class.

Let $0<\alpha\leq 1$. Then, by $ULip^N(\alpha)$ we denote the class of all functions f belonging to $BUC_{\log}\left(\mathbb{R}_+^N\right)$ for which

$$\|\tau_{\mathbf{t}}(f)-f\|_{\infty} = O\left(|\log \mathbf{t}|^{\alpha}\right), \text{ as } |\mathbf{1}-\mathbf{t}|\to 0.$$

Theorem 3.3. *Let $\mathcal{A}=\{[a_{nk}^{\nu}]\}$ be a nonnegative regular summability method and let $\{L_k\}$ be a sequence of kernels such that $\sup_{k\in\mathbb{N}}\|L_k\|_{L_\mu^1}=M<\infty$ for some $M>0$, and let $\alpha\in(0,1]$. Assume that (2.3), (2.4), (2.5) hold. Assume further that $\{\gamma_k\}$ is a null sequence of positive real numbers satisfying that*

$$\begin{gathered} \|G_k\|_J \leq \gamma_k \\ \text{(for all } k\in\mathbb{N} \text{ and for every bounded interval } J\subset\mathbb{R}\text{)} \end{gathered} \tag{3.1}$$

and

$$\sum_{k=1}^{\infty} a_{nk}^{\nu}\gamma_k = O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \nu\text{)}. \tag{3.2}$$

Then, for every $f\in ULip^N(\alpha)$,

$$\|\mathcal{T}_{n,\nu}(f)-f\|_{\infty} = O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \nu\text{)}.$$

Proof. As in the proof of the Theorem 3.2, it is clear that

$$\begin{aligned}\left\|\mathcal{T}_{n,\nu}(f)-f\right\|_{\infty} &\leq \sum_{k=1}^{\infty} a_{nk}^{\nu}\left\|G_k\right\|_J \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \frac{\mathbf{dt}}{\langle \mathbf{t}\rangle} \\ &+ \sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} \left|L_k(\mathbf{t})\right| \left\|\tau_{\mathbf{t}}(f)-f\right\|_{\infty} \frac{\mathbf{dt}}{\langle \mathbf{t}\rangle} \\ &+ \|f\|_{\infty}\left|\sum_{k=1}^{\infty} a_{nk}^{\nu} \int\limits_{\mathbb{R}_+^N} L_k(\mathbf{t}) \frac{\mathbf{dt}}{\langle \mathbf{t}\rangle}-1\right| \\ &:= J_1(n,\nu)+J_2(n,\nu)+J_3(n,\nu)\end{aligned}$$

holds. From (3.1), we get

$$J_1(n,\nu) \leq M \sum_{k=1}^{\infty} a_{nk}^{\nu}\gamma_k.$$

Therefore, using this and also considering the assumptions (2.3), (2.4), (2.5), (3.2) we easily arrive to the following:

$$J_i(n,\nu)=O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \nu).$$

for each $i=1,2,3$, which completes the proof. □

4. An Application and Graphical Illustrations

In this section, using the operators (1.1), we approximate to the function

$$f(x,y) := \frac{1}{\sqrt{2}}\left|\left(\sin\left(\log x\right), \sin\left(\log y\right)\right)\right| \tag{4.1}$$

defined on $\mathbb{R}_+^2$.

Using the inverse triangle inequality, one can see that

$$\begin{aligned}\left|f(x,y)-f(u,v)\right| &= \frac{1}{\sqrt{2}}\left|\left|\left(\sin\left(\log x\right),\sin\left(\log y\right)\right)\right|-\left|\left(\sin\left(\log u\right),\sin\left(\log v\right)\right)\right|\right| \\ &\leq \frac{1}{\sqrt{2}}\left|\left(\sin\left(\log x\right)-\sin\left(\log u\right),\sin\left(\log y\right)-\sin\left(\log v\right)\right)\right| \\ &\leq \frac{1}{\sqrt{2}}\left|\left(\log x-\log u,\log y-\log v\right)\right|.\end{aligned}$$

Then we immediately get $f \in BUC_{\log}\left(\mathbb{R}_+^2\right)$.

Define the functions $L_k:\mathbb{R}_+^2\to\mathbb{R}$ and $H_k:\mathbb{R}\to\mathbb{R}$ as follows, respectively:

$$L_k(s,t) := \begin{cases} \dfrac{4k^2}{\pi}\left(\left(-1\right)^k+1\right)st, & \text{if } (s-1)^2+(t-1)^2 \leq \dfrac{1}{4k^2} \\ 0, & \text{otherwise}\end{cases} \tag{4.2}$$

and

$$H_k(u) := \begin{cases} u+e^{u/k}-1, & \text{if } 0\leq u<1 \\ u+e^{1/(uk)}-1, & \text{if } u\geq 1,\end{cases} \tag{4.3}$$

where we extend $H_k(u)$ in the odd-way for $u<0$.

Consider the Cesàro matrix summability $\mathcal{A} = \{C_1\} = \{(c_{nk})\}$ given by (see [18])

$$c_{nk} = \begin{cases} \dfrac{1}{n}, & 1 \leq k \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Now take $\alpha = \frac{1}{2}$. Then, we show that all conditions $(i) - (iv)$ and $(2.3) - (2.7)$ hold. By the definition of L_k, we get

$$\begin{aligned} \sum_{k=1}^{n} c_{nk} \|L_k\|_{L^1_\mu} &= \frac{4}{n\pi} \sum_{k=1}^{n} k^2 \left((-1)^k + 1\right) \iint\limits_{(1-s)^2+(1-t)^2 \leq \frac{1}{4k^2}} dsdt \\ &= \frac{1}{n} \sum_{k=1}^{n} \left((-1)^k + 1\right) \\ &= \begin{cases} 1, & \text{if } n \text{ is even} \\ 1 - \frac{1}{n}, & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

which immediately implies (i). Using the last equality, we may also write that

$$\left| \sum_{k=1}^{n} c_{nk} \int\limits_{\mathbb{R}^2_+} L_k(s,t) \frac{dsdt}{st} - 1 \right| \leq \frac{1}{n} \leq \frac{1}{\sqrt{n}},$$

which gives conditions (ii) and (2.3) for $\alpha = \frac{1}{2}$. Now, for any fixed $\delta > 0$, we get

$$\begin{aligned} &\sum_{k=1}^{n} c_{nk} \iint\limits_{(1-s)^2+(1-t)^2<\delta^2} |L_k(s,t)| \left(\log^2 s + \log^2 t\right)^{1/4} \frac{dsdt}{st} \\ &\leq \frac{1}{n} \sum_{k=1}^{n} \int\limits_{\mathbb{R}^2_+} |L_k(s,t)| \left(\log^2 s + \log^2 t\right)^{1/4} \frac{dsdt}{st} \\ &= \frac{4}{n\pi} \sum_{k=1}^{n} k^2 \left((-1)^k + 1\right) \iint\limits_{(1-s)^2+(1-t)^2 \leq \frac{1}{4k^2}} \left(\log^2 s + \log^2 t\right)^{1/4} dsdt. \end{aligned}$$

Since $(1-s)^2 + (1-t)^2 \leq \frac{1}{4k^2} \leq \frac{1}{4}$, we observe that $\frac{1}{2} \leq s \leq \frac{3}{2}$ which implies $|\log s| \leq |1-s|$. Similarly, $|\log t| \leq |1-t|$ due to $\frac{1}{2} \leq t \leq \frac{3}{2}$. Hence,

$$\log^2 s + \log^2 t \leq (1-s)^2 + (1-t)^2$$

holds. Then, it follows from the last inequality that

$$
\begin{aligned}
&\sum_{k=1}^{n} c_{nk} \iint\limits_{(1-s)^2+(1-t)^2<\delta^2} |L_k(s,t)| \left(\log^2 s + \log^2 t\right)^{1/4} \frac{dsdt}{st} \\
\leq\ & \frac{4}{n\pi} \sum_{k=1}^{n} k^2 \left((-1)^k + 1\right) \iint\limits_{(1-s)^2+(1-t)^2\leq\frac{1}{4k^2}} \left((1-s)^2 + (1-t)^2\right)^{1/4} dsdt \\
=\ & \frac{4}{n\pi} \sum_{k=1}^{n} k^2 \left((-1)^k + 1\right) \int\limits_0^{2\pi} \int\limits_0^{1/2k} r^{3/2} drd\theta \\
\leq\ & \frac{2}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{k}}.
\end{aligned}
$$

Since $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n}$, we may write that

$$
\sum_{k=1}^{n} c_{nk} \iint\limits_{(1-s)^2+(1-t)^2<\delta^2} |L_k(s,t)| \left(\log^2 s + \log^2 t\right)^{1/4} \frac{dsdt}{st} \leq \frac{4}{\sqrt{n}},
$$

which guarantees that condition (2.4) is satisfied for $\alpha = \frac{1}{2}$. By using a similar way, for any fixed $\delta > 0$ and for any sufficiently large n, we observe that

$$
\begin{aligned}
&\sum_{k=1}^{n} c_{nk} \iint\limits_{(1-s)^2+(1-t)^2\geq\delta^2} |L_k(s,t)| \frac{dsdt}{st} \\
=\ & \frac{1}{n} \sum_{k=1}^{\left[\frac{1}{2\delta}\right]} \iint\limits_{(1-s)^2+(1-t)^2\geq\delta^2} |L_k(s,t)| \frac{dsdt}{st} \\
& + \frac{1}{n} \sum_{k=\left[\frac{1}{2\delta}\right]+1}^{n} \iint\limits_{(1-s)^2+(1-t)^2\geq\delta^2} |L_k(s,t)| \frac{dsdt}{st}
\end{aligned}
$$

with the convention that the empty summation is zero, where $[\cdot]$ means the integer part. From (4.2) , the second summation on the right-hand side of the last equality must be zero. Since

$$
\begin{aligned}
&\sum_{k=1}^{\left[\frac{1}{2\delta}\right]} \iint\limits_{(1-s)^2+(1-t)^2\geq\delta^2} |L_k(s,t)| \frac{dsdt}{st} \\
=\ & \frac{4}{\pi} \sum_{k=1}^{\left[\frac{1}{2\delta}\right]} k^2 \left((-1)^k + 1\right) \iint\limits_{\delta^2\leq(1-s)^2+(1-t)^2\leq\frac{1}{4k^2}} dsdt \\
\leq\ & \frac{8}{\pi} \sum_{k=1}^{\left[\frac{1}{2\delta}\right]} k^2 \int\limits_0^{2\pi} \int\limits_\delta^{1/2k} rdrd\theta \\
\leq\ & \frac{1}{\delta}
\end{aligned}
$$

we obtain that

$$\sum_{k=1}^{n} c_{nk} \iint\limits_{(1-s)^2+(1-t)^2 \geq \delta^2} |L_k(s,t)| \frac{dsdt}{st} \leq \frac{1}{\delta n} \leq \frac{1}{\delta\sqrt{n}},$$

which yields conditions (iii) and (2.4) for $\alpha = \frac{1}{2}$.

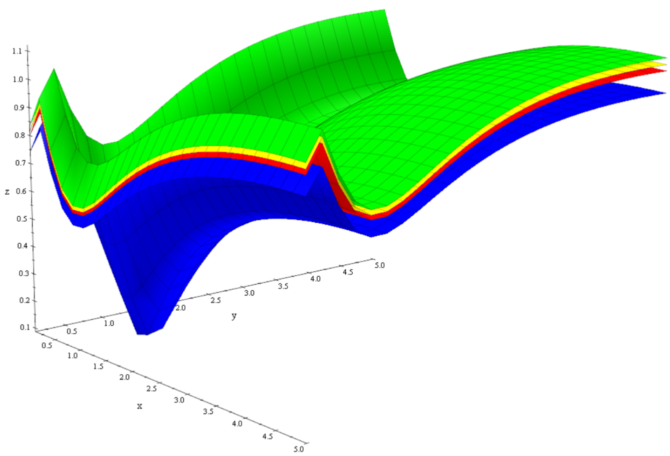

FIGURE 1. Approximation to f by means of $\mathcal{T}_{n,\nu}(f)$ for odd values of $n = 15, 23, 35$.

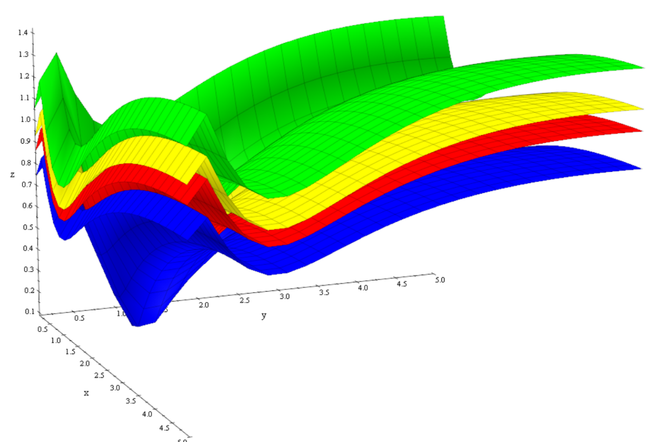

FIGURE 2. Approximation to f by means of $\mathcal{T}_{n,\nu}(f)$ for even values of $n = 4, 10, 20$.

Finally, we know from [6] that the function H_k in (4.3) satisfies assumptions (iv) and $(iv)'$; and also for every bounded interval $J \subset \mathbb{R}$

$$\frac{V_J[G_k]}{m(J)} \leq \frac{2e}{k}$$

holds. Hence, defining the null sequence $\{\beta_k\} = \{\frac{2e}{\sqrt{k}}\}$, one can get assumptions (2.6) and (2.7).

As a result, it is possible to approximate to the function f defined by (4.1) by means of the sequence $\{\mathcal{T}_n(f)\}$ based on (4.2) and (4.3). This approximation is indicated in Figures 1 and 2 for different values of n, where the bottom surface coloured with blue shows the graph of f.

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