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SOME DOUBLE BINOMIAL SUMS RELATED WITH THE FIBONACCI, PELL AND GENERALIZED ORDER-k FIBONACCI NUMBERS

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ABSTRACT. We consider some double binomial sums related to the Fibonacci and Pell numbers and a multiple binomial sum related to the generalized order-k Fibonacci numbers. The Lagrange-Bürmann formula and other well-known techniques are used to prove them.

1. Introduction. The generating function of the Fibonacci numbers F_n is

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}.$$

Similarly, the generating function of the Pell numbers P_n is

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-x^2}.$$

The generalized order-k Fibonacci numbers $f_n^{(k)}$ are defined by

$$f_n^{(k)} = \sum_{i=1}^{k} f_{n-i}^{(k)} \quad \text{for } n > k,$$

with initial conditions $f_j^{(k)} = 2^{j-1}$ for $1 \le j \le k$.

For example, when $k=3$, the generalized Fibonacci numbers $f_n^{(3)}$ are reduced to the Tribonacci numbers T_n defined by

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$$T_n = T_{n-1} + T_{n-2} + T_{n-3},$$

with $T_1 = 1$, $T_2 = 2$ and $T_3 = 4$, for $n > 3$.

For these number sequences, we recall the combinatorial representations due to [**2, 3, 5**]:

$$\sum_{i=1}^{n} \binom{n-i}{i-1} = F_n, \tag{1.1}$$

$$\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} 2^i = P_n, \tag{1.2}$$

$$\sum_{\substack{0 \le 1 \\ j \le n}} \binom{n-i}{j} \binom{n-j}{i} = F_{2n+3}. \tag{1.3}$$

Among the formulas (1.1)–(1.3), the last formula seems to be different from the first above since it includes double sums, see [**2**]. The authors of the above-cited papers use a combinatorial approach to prove these results. For many similar identities, we refer to [**6**].

In this paper, we shall derive some new double binomial sums related with the Fibonacci, Pell and generalized order-k Fibonacci numbers and then use the Lagrange-Bürmann formula and other well-known techniques to prove them.

The Lagrange-Bürmann formula is a very useful tool if one knows a series expansion for $y(x)$ but would like to obtain the series for x in terms of y. We recall the formula (for details, see [**1, 4**]): Suppose a series for y in powers of x is required when $y = x\Phi(y)$. Assume that Φ is analytic in a neighborhood of $y = 0$ with $\Phi(0) \neq 0$. Then

$$x = y/\Phi(y) = \sum_{n=1}^{\infty} a_n y^n, \quad a_1 \neq 0.$$

Then the two (equivalent) versions of the Lagrange(-Bürmann) inversion formula can be written as

$$F(y) = F(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dy^{n-1}} \big(F'(y) \Phi^n(y) \big) \right]_{x=0}$$

or

$$\frac{F(y)}{1 - x\Phi'(y)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left[\frac{d^n}{dy^n} \big(F(y)\Phi^n(y) \big) \right]_{x=0}.$$

We would like to rephrase this using the notation of the "coefficient-of" operator:

$$\frac{F(y)}{1 - x\Phi'(y)} = \sum_{n=0}^{\infty} [y^n] \big(F(y)\Phi^n(y) \big) \cdot x^n;$$

we will use it in this form.

2. Double binomial sums. We start with a result related to Fibonacci numbers:

Theorem 1. *For* $n > 0$,

$$F_{4n-1} = \sum_{0 \le i,j \le n} \binom{n+i}{2j} \binom{n+j}{2i}.$$

Proof. We start from

$$[y^{2j}](1+y)^{n+i} = \binom{n+i}{2j}$$

and compute

$$\begin{aligned} S &= \sum_{i=0}^{n} (1+y)^{n+i} \binom{n+j}{2i} \\ &= \sum_{i \ge 0} (1+y)^{n+i/2} \binom{n+j}{i} \frac{1+(-1)^i}{2} \\ &= \left[\left(1+\sqrt{1+y}\right)^{j+n} + \left(1-\sqrt{1+y}\right)^{j+n} \right] \frac{(1+y)^n}{2}; \end{aligned}$$

here the desired sum takes the form:

$$\sum_{j=0}^{n} [y^{2j}] \left[\left(1+\sqrt{1+y}\right)^{j+n} + \left(1-\sqrt{1+y}\right)^{j+n} \right] \frac{(1+y)^n}{2}$$

$$
\begin{aligned}
&= \sum_{j\ge 0}[y^{2j}]\left(1+\sqrt{1+y}\right)^{j+n}\frac{(1+y)^n}{2}\\
&\quad+\sum_{j\ge 0}[y^{2j}]\left(1-\sqrt{1+y}\right)^{j+n}\frac{(1+y)^n}{2}\\
&= \sum_{j\ge 0}[y^{j}]\left(1+\sqrt{1+y}\right)^{j/2+n}\frac{(1+y)^n}{2}\frac{1+(-1)^j}{2}\\
&\quad+\sum_{j\ge 0}[y^{2j}]\left(1-\sqrt{1+y}\right)^{j+n}\frac{(1+y)^n}{2}.
\end{aligned}
$$

Let us consider the first sum:

$$\sum_{j\ge 0}[y^j]\left(1+\sqrt{1+y}\right)^{j/2+n}(1+y)^n.$$

This is of the form

$$\sum_{j\ge 0}[y^j]F(y)\Phi(y)^j$$

with

$$F(y)=\left(1+\sqrt{1+y}\right)^n(1+y)^n \quad \text{and} \quad \Phi(y)=\sqrt{1+\sqrt{1+y}}.$$

The Lagrange-Bürmann formula can now be applied to this sum. The general formula is given by:

$$\sum_{j\ge 0}[y^j]F(y)\Phi(y)^j\cdot x^j=\frac{F(y)}{1-x\Phi'(y)}.$$

We need the instance $x=1$ here, and the variables x and y are linked via $y=x\Phi(y)$. Notice that $\Phi(y)$ must be a power series in y with a constant term different from zero. Therefore, by the solution of $y=\Phi(y)$, we find $y=\alpha=(1+\sqrt{5})/2$, and so

$$
\begin{aligned}
y&=\frac{1+\sqrt{5}}{2}, &\qquad F(\alpha)&=\left(\frac{7+3\sqrt{5}}{2}\right)^n,\\
\Phi'(\alpha)&=\frac{3-\sqrt{5}}{8}, &\qquad \frac{1}{1-\Phi'(\alpha)}&=2\left(1-\frac{1}{\sqrt{5}}\right).
\end{aligned}
$$

So our evaluation is

$$2\left(1-\frac{1}{\sqrt{5}}\right)\left(\frac{7+3\sqrt{5}}{2}\right)^{n}.$$

The second term is

$$\sum_{j\geq 0}[y^j]\left(1+\sqrt{1+y}\right)^{j/2+n}(1+y)^n(-1)^j.$$

This is the instance $x=-1$, which translates to $y=-1$, and so the second term is

$$\frac{F(-1)}{1+\Phi'(-1)}=0.$$

The last sum is

$$\begin{aligned}\sum_{j\geq 0}[y^{2j}]&\left(1-\sqrt{1+y}\right)^{j+n}(1+y)^n\\ &=\sum_{j\geq 0}[y^{2j}]y^{j+n}\left(\frac{1-\sqrt{1+y}}{y}\right)^{j+n}(1+y)^n\\ &=\sum_{j\geq 0}[y^{j}]y^{n}\left(\frac{1-\sqrt{1+y}}{y}\right)^{j+n}(1+y)^n.\end{aligned}$$

This is again of the form

$$\sum_{j\geq 0}[y^j]F(y)\Phi(y)^j$$

with

$$F(y)=\left(1-\sqrt{1+y}\right)^{n}(1+y)^n$$

and

$$\Phi(y)=\frac{1-\sqrt{1+y}}{y}.$$

We need the instance $x = 1$ here, and the link is

$$y = x\left(\frac{1-\sqrt{1+y}}{y}\right).$$

By the solution of the last equation, we find $y = \beta$ where $\beta = (1-\sqrt{5})/2$, and so we write

$$y = \beta = \frac{1-\sqrt{5}}{2},$$
$$F(\beta) = \left(\frac{7-3\sqrt{5}}{2}\right)^n$$

and

$$\frac{1}{1-\Phi'(\alpha)} = 1 + \frac{1}{\sqrt{5}}.$$

So our evaluation is

$$\left(1+\frac{1}{\sqrt{5}}\right)\left(\frac{7-3\sqrt{5}}{2}\right)^n.$$

Altogether,

$$\left[\left(1-\frac{1}{\sqrt{5}}\right)\left(\frac{7+3\sqrt{5}}{2}\right)^n + \left(1+\frac{1}{\sqrt{5}}\right)\left(\frac{7-3\sqrt{5}}{2}\right)^n\right]\frac{1}{2}$$
$$= \frac{\alpha^{4n-1}-\beta^{4n-1}}{\sqrt{5}} = F_{4n-1},$$

as desired. □

Theorem 2. *For $n > 0$,*

$$F_{4n+1} = \sum_{1\le i,j\le n+1} \binom{n+i}{2j-1}\binom{n+j}{2i-1}.$$

Proof. Since

$$\left[y^{2j-1}\right](1+y)^{n+i} = \binom{n+i}{2j-1}$$

and

$$
\begin{aligned}
S &= \sum_{i=1}^{n+1} (1+y)^{n+i} \binom{n+j}{2i-1} \\
&= \sum_{i\ge 0} (1+y)^{n+(i+1)/2} \binom{n+j}{i} \frac{1-(-1)^i}{2} \\
&= \left[\left(1+\sqrt{1+y}\right)^{j+n} - \left(1-\sqrt{1+y}\right)^{j+n} \right] \frac{(1+y)^{n+1/2}}{2};
\end{aligned}
$$

here the desired sum takes the form:

$$
\begin{aligned}
&\sum_{j=1}^{n+1} [y^{2j-1}] \left[\left(1+\sqrt{1+y}\right)^{j+n} - \left(1-\sqrt{1+y}\right)^{j+n} \right] \frac{(1+y)^{n+1/2}}{2} \\
&\quad = \sum_{j\ge 1} [y^{2j-1}] \left(1+\sqrt{1+y}\right)^{j+n} \frac{(1+y)^{n+1/2}}{2} \\
&\qquad - \sum_{j\ge 1} [y^{2j-1}] \left(1-\sqrt{1+y}\right)^{j+n} \frac{(1+y)^{n+1/2}}{2} \\
&\quad = \sum_{j\ge 0} [y^{j}] \left[\left(1+\sqrt{1+y}\right)^{j/2+n+1/2} \right] \frac{(1+y)^{n+1/2}}{2} \frac{1-(-1)^j}{2} \\
&\qquad - \sum_{j\ge 1} [y^{2j-1}] \left(1-\sqrt{1+y}\right)^{j+n} \frac{(1+y)^{n+1/2}}{2}.
\end{aligned}
$$

Let us start with one term in the above sum:

$$
\sum_{j\ge 0} [y^{j}] \left(1+\sqrt{1+y}\right)^{j/2+n+1/2} (1+y)^{n+1/2}.
$$

This is of the form:

$$
\sum_{j\ge 0} [y^j] F(y) \Phi(y)^j,
$$

with

$$
F(y) = \left(1+\sqrt{1+y}\right)^{n+1/2} (1+y)^{n+1/2}
$$

and

$$\Phi(y) = \sqrt{1+\sqrt{1+y}}.$$

This is the instance $x = 1$, which, by $\alpha = (1+\sqrt{5})/2$, translates to

$$y = \frac{1+\sqrt{5}}{2}, \qquad F(\alpha) = \alpha^{4n+2}$$

and

$$\Phi'(\alpha) = \frac{3-\sqrt{5}}{8}, \qquad \frac{1}{1-\Phi'(\alpha)} = 2\Big(1-\frac{1}{\sqrt{5}}\Big).$$

So our evaluation is:

$$2\Big(1-\frac{1}{\sqrt{5}}\Big)\alpha^{4n+2}.$$

The second term is:

$$\sum_{j\ge 0}[y^j]\left(1+\sqrt{1+y}\right)^{j/2+n+1/2}(1+y)^{n+1/2}(-1)^j.$$

This is the instance $x = -1$, which translates to $y = -1$ and so the second term is:

$$\frac{F(-1)}{1+\Phi'(-1)} = 0.$$

Finally the last term is of the form:

$$\begin{aligned}\sum_{j\ge 1}&[y^{2j-1}]\left(1-\sqrt{1+y}\right)^{j+n}(1+y)^{n+1/2}\\ &= \sum_{j\ge 1}[y^{2j-1}]y^{j+n}\left(\frac{1-\sqrt{1+y}}{y}\right)^{j+n}(1+y)^{n+1/2}\\ &= \sum_{j\ge 0}[y^{j}]y^{n+1}\left(\frac{1-\sqrt{1+y}}{y}\right)^{j+n}(1+y)^{n+1/2}.\end{aligned}$$

This is of the form:

$$\sum_{j\ge 0}[y^j]F(y)\Phi(y)^j$$

with

$$F(y) = \left(1 - \sqrt{1+y}\right)^n (1+y)^{n+\frac{1}{2}} y$$

and

$$\Phi(y) = \frac{1 - \sqrt{1+y}}{y}.$$

This is the instance $x = 1$, which translates to $y = \beta = (1 - \sqrt{5})/2$. Thus,

$$\begin{aligned} F(\beta) &= -\beta^{4n+2}, \\ \Phi'(\beta) &= -\frac{1-\sqrt{5}}{4}, \\ \frac{F(\beta)}{1 - \Phi'(\beta)} &= -\left(1 + \frac{1}{\sqrt{5}}\right)\beta^{4n+2}. \end{aligned}$$

So our evaluation is:

$$\left[\left(1 - \frac{1}{\sqrt{5}}\right)\alpha^{4n+2} + \left(1 + \frac{1}{\sqrt{5}}\right)\beta^{4n+2}\right]\frac{1}{2} = F_{4n+1},$$

as claimed. □

Theorem 3. *For $n > 0$,*

$$\begin{aligned} F_{4n} &= \sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n+i}{2j-1}\binom{n+j}{2i}, \\ F_{4n-3} &= \sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n+i}{2j+1}\binom{n+j}{2i+1}. \end{aligned}$$

Again by using the Lagrange-Bürmann formula, Theorem 3 can be similarly proved.

Theorem 4. *For $n > 0$,*

$$\frac{F_{2n+2}+F_{n+1}}{2} = \sum_{0\le i,j\le n} \binom{n-i}{2j}\binom{n-2j}{i}.$$

Proof. First, we replace i by $n-i$ and get

$$\sum_{0\le 2j\le i\le n} \binom{i}{2j}\binom{n-2j}{i-2j}.$$

Now we compute the generating function of it:

$$\begin{aligned}
\sum_{n\ge 0} z^n \sum_{0\le 2j\le i\le n} \binom{i}{2j}\binom{n-2j}{i-2j}
&= \sum_{0\le 2j\le i} \binom{i}{2j} \frac{z^i}{(1-z)^{i+1-2j}} \\
&= \sum_{j\ge 0} \frac{z^{2j}(1-z)^{2j}}{(1-2z)^{1+2j}} = \frac{1-2z}{(1-z-z^2)(1-3z+z^2)} \\
&= \frac{1}{2}\frac{1}{1-z-z^2} + \frac{1}{2}\frac{1}{1-3z-z^2},
\end{aligned}$$

which is the generating function of the numbers $(F_{2n+2}+F_{n+1})/2$. □

The following results are similar:

Theorem 5. *For $n > 0$,*

$$F_{2n} = \sum_{i=1}^{n}\sum_{j=1}^{n} \binom{n-i}{j-1}\binom{n-j}{i-1},$$

$$F_{2n-1} = \sum_{0\le j\le i\le n} \binom{n}{i-j}\binom{n-i}{j}.$$

Theorem 6. *For $n > 0$,*

$$F_{2n}+1 = \sum_{i=0}^{n} F_{2i-1} = \sum_{0\le i\le j\le n} \binom{n-i}{j}\binom{j}{2i}. \tag{2.1}$$

Proof. Multiplying the right hand side of (2.1) by z^n and summing over n, we get

$$\begin{aligned} S &= \sum_{n\geq 0} z^n \sum_{0\leq i\leq j\leq n} \binom{n-i}{j}\binom{j}{2i} \\ &= \sum_{0\leq i\leq j} \sum_{h\geq 0} z^{h+i+j} \binom{h+j}{j}\binom{j}{2i} \\ &= \sum_{0\leq i\leq j} \binom{j}{2i} z^{i+j} \sum_{h\geq 0} z^h \binom{h+j}{j} \\ &= \sum_{0\leq 2i\leq j} \binom{j}{2i} z^{i+j} \frac{1}{(1-z)^{j+1}} \\ &= \sum_{i\geq 0} \frac{z^{3i}}{(1-2z)^{2i+1}} = \frac{1-2z}{(1-z)(1-3z+z^2)} \\ &= \frac{z}{1-3z+z^2} + \frac{1}{1-z}, \end{aligned}$$

which is the generating function of the numbers $F_{2n}+1$. □

For the Pell numbers, we give the following result:

Theorem 7. *For* $n \geq 0$,

$$P_{n+1} = \sum_{0\leq i\leq j\leq n} \binom{n-i}{j}\binom{j}{i}. \tag{2.2}$$

Proof. Multiplying the right hand side of (2.2) by z^n and summing over n, we get

$$\begin{aligned} S &= \sum_{n\geq 0} z^n \sum_{0\leq i\leq j\leq n} \binom{n-i}{j}\binom{j}{i} \\ &= \sum_{0\leq i\leq j} \sum_{h\geq 0} z^{h+i+j} \binom{h+j}{j}\binom{j}{i} \end{aligned}$$

$$= \sum_{0\le i\le j} \binom{j}{i} z^{i+j} \sum_{h\ge 0} z^h \binom{h+j}{j}$$
$$= \sum_{0\le i\le j} \binom{j}{i} z^{i+j} \frac{1}{(1-z)^{j+1}}$$
$$= \sum_{0\le i\le j} \frac{z^j}{(1-z)^{j+1}} \binom{j}{i} z^i = \sum_{j\ge 0} \frac{z^j}{(1-z)^{j+1}} (1+z)^j$$
$$= \frac{1}{1-z} \frac{1}{1-z(1+z)/(1-z)} = \frac{1}{1-2z-z^2}.$$

This is the generating function of the numbers P_{n+1}. □

Now we give a double sum for the Tribonacci numbers:

Theorem 8. *For $n \ge 0$,*

$$T_n = \sum_{0\le j\le i\le n} \binom{n-i}{i-j} \binom{i-j}{j}.$$

Proof. Consider

$$\sum_{n\ge 0} T_n z^n = \sum_{0\le j\le i\le n} z^n \binom{n-i}{i-j} \binom{i-j}{j}$$
$$= \sum_{0\le j\le i} z^i \binom{i-j}{j} \sum_{h\ge 0} z^h \binom{h}{i-j}$$
$$= \sum_{0\le j\le i} z^i \binom{i-j}{j} \frac{z^{i-j}}{(1-z)^{i-j+1}}$$
$$= \sum_{j\ge 0} \sum_{h\ge 0} z^{h+j} \binom{h}{j} \frac{z^h}{(1-z)^{h+1}}.$$

Let $t = z^2/(1-z)$, and we continue with

$$\sum_{n\ge 0} T_n z^n = \frac{1}{1-z} \sum_{0\le j} z^j \sum_{h\ge 0} \binom{h}{j} t^h = \frac{1}{1-z} \sum_{0\le j} z^j \frac{t^j}{(1-t)^{j+1}}$$

$$= \frac{1}{1-z}\frac{1}{1-t}\frac{1}{1-zt/(1-t)} = \frac{1}{1-z}\frac{1}{1-t-zt}$$
$$= \frac{1}{1-z}\frac{1}{1-z^2/(1-z)-z^3/(1-z)} = \frac{1}{1-z-z^2-z^3},$$

which is the generating function of the Tribonacci numbers, as expected. So the proof is complete. □

By using the same proof method as in Theorem 8, we get a more general result:

Theorem 9. *For $n > 0$,*

$$f_n^{(k)} = \sum_{0 \le i_k \le \cdots \le i_1 \le n} \binom{n-i_1}{i_1-i_2}\binom{i_1-i_2}{i_2-i_3}\cdots\binom{i_{k-1}-i_k}{i_k},$$

where $f_n^{(k)}$ is the nth generalized order-k Fibonacci number.

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