SUMS OF PRODUCTS OF GENERALIZED FIBONACCI AND LUCAS NUMBERS

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Abstract. Sums of products of a fixed number of Fibonacci(-type) numbers can be computed automatically. This extends, at least in principle, various results about two factors that appeared in the literature. More general results (with an arbitrary number of factors) where then guessed and eventually proved by traditional ("human") methods.

1. Introduction

The authors of [4, 5, 7, 8, 10, 11] where all concerned with the evaluation of

$$\sum_{k=1}^{n} u_{a+bk} v_{c+dk},$$

where u_k, v_k denote either Fibonacci or Lucas numbers or generalizations of them (recursions of second order, with some parameters).

In this paper, we evaluate much more general sums, with an arbitrary number of factors. Such evaluations can be done by a computer: What is important is the existence of a *Binet formula* for the numbers involved; all that a computer must do is to sum some geometric series. We will describe the procedure for *Fibonacci numbers*, but that is no restriction whatsoever; it is merely done to avoid cluttering the notation.

We have the classical representation

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{with} \quad \alpha, \beta = \frac{1 \pm \sqrt{5}}{2};$$

note that $\alpha\beta = -1$.

If it is desired, the outcome of the computers computation, which is given in terms of powers of α and β, can be given in terms of Fibonacci numbers. This is straight-forward, using the formulæ

$$\alpha^n = \alpha F_n + F_{n-1}, \qquad \beta^n = -\alpha F_n + F_{n+1}.$$

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Typically, the outcome of these replacements is not æsthetically pleasing, and simplifications can be done using the standard recursion for Fibonacci numbers.

To give the reader an idea, we can compute with this approach that

$$\sum_{k=0}^{n} F_{4k}F_{4k+1} = -\frac{n}{5} - \frac{1}{3} - \frac{41}{75}F_{8n+10} + \frac{41}{75}F_{8n+18} + \frac{76}{225}F_{8n+11} - \frac{76}{225}F_{8n+19}.$$

The "best" expression is however

$$\sum_{k=0}^{n} F_{4k}F_{4k+1} = -\frac{n}{5} - \frac{1}{3} + \frac{1}{15}F_{8n+5}.$$

We can find this with a computer, since, by using the ordinary recursion for Fibonacci numbers, we can always find a representation

$$\sum_{k=0}^{n} F_{4k}F_{4k+1} = -\frac{n}{5} - \frac{1}{3} + xF_{8n+m} + yF_{8n+m+1},$$

and we may search for m in a small range, so that $y = 0$. Of course, this is not guaranteed, but if it exists, we find it.

We want to convince the reader that this approach is not restricted to Fibonacci numbers, but is fully general. To demonstrate this, the following example should be sufficient:

Let

$$U_n = pU_{n-1} + U_{n-1} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$V_n = pV_{n-1} + V_{n-2} = \alpha^n + \beta^n$$

with

$$\alpha = \frac{p + \sqrt{p^2+4}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2+4}}{2},$$

then our approach evaluates

$$S := \sum_{k=0}^{n} U_{3k+7}U_{2k+3}V_{k+2}$$

as

$$\begin{aligned} S = \frac{\alpha^7}{(\alpha^4+1)(\alpha^6-1)(\alpha^2+1)^2}\big[&V_{6n+17} + V_{6n+13} + V_{4n+13} - V_{4n+7} \\ &+ V_{2n+11} + V_{2n+9} + 2V_{2n+7} + V_{2n+5} + V_{2n+3}\big] \\ - \frac{\alpha^{12}+3\alpha^6+1}{\alpha^4(\alpha^2+1)^2}&. \end{aligned}$$

Note that

$$\frac{\alpha^7}{(\alpha^4+1)(\alpha^6-1)(\alpha^2+1)^2} = \frac{1}{p(p^2+2)(p^2+3)(p^2+4)}$$

and

$$\frac{\alpha^{12}+3\alpha^6+1}{\alpha^4(\alpha^2+1)^2} = \frac{p^6+6p^4+9p^2+5}{p^2+4}.$$

Using this approach, we were hunting for attractive results by computing many sums with various parameters involved and spotting patterns. Eventually, proving the identities that we collected in the following section, required the manipulation of certain expressions using the q-binomial theorem and similar tools. We allowed an arbitrary number of factors, and this is typically hard or even impossible for a computer algebra system, and henceforth we provided "traditional" (as opposed to "mechanical") proofs.

2. New Fibonacci summations

We get explicit evaluations of

$$X_n^{(d)} = \sum_{k=0}^{n} \prod_{j=0}^{d-1} F_{tk+\lambda j},$$

where t is an even number. We do not make the parameters λ and t part of the notation, for better readibility. Cases must be distinguished between $t \equiv 0 \bmod 4$ and $t \equiv 2 \bmod 4$, and d even or odd. Thus we get four theorems.

Theorem 1. *For $t \equiv 0 \bmod 4$ and even d, we have*

$$X_n^{(d)} = \sum_{j=1}^{d/2} F_{2tnj+((d-1)\lambda+t)j}\, a_{d,j} + c_1 \cdot n + c_2,$$

with

$$a_{d,j} = \frac{1}{5^{d/2}F_{tj}} \left\{ \begin{matrix} d \\ \frac{d}{2}-j \end{matrix} \right\}_\lambda \times \begin{cases} (-1)^{\frac{d}{2}-j} & \text{if } \lambda \text{ is even,} \\ (-1)^{\frac{(\frac{d}{2}-j)(\frac{d}{2}-j+1)}{2}} & \text{if } \lambda \text{ is odd.} \end{cases}$$

Here and in the following we use (for more details see [2, 3]*)*

$$\left\{ \begin{matrix} m \\ k \end{matrix} \right\}_\lambda = \frac{\prod_{j=1}^{m} F_{j\lambda}}{\prod_{j=1}^{k} F_{j\lambda} \cdot \prod_{j=1}^{m-k} F_{j\lambda}}.$$

REMARK. We do not give an explicit expression for the constants c_1 and c_2, since they can be recovered from the rest: Assume that we have an identity

$$A(n) = B(n) + c_1 \cdot n + c_2,$$

then $c_2 = A(0) - B(0)$ and $c_1 = A(1) - A(0) - B(1) + B(0)$. This remark applies also to the following examples.

Theorem 2. *For $t \equiv 0 \bmod 4$ and odd d, we have*

$$X_n^{(d)} = \sum_{j=0}^{(d-1)/2} L_{tn(2j+1)+((d-1)\lambda+t)\frac{2j+1}{2}} a_{d,j} + c_1 \cdot n + c_2,$$

where

$$a_{d,j} = \frac{1}{5^{(d+1)/2}} F_{\frac{t(2j+1)}{2}} \left\{ \begin{matrix} d \\ \frac{d-1}{2} - j \end{matrix} \right\}_\lambda \times \begin{cases} (-1)^{\frac{d-1}{2}-j} & \text{if } \lambda \text{ is even,} \\ (-1)^{\frac{(\frac{d-1}{2}-j)(\frac{d+1}{2}-j)}{2}} & \text{if } \lambda \text{ is odd.} \end{cases}$$

Here and in the following, we use *Lucas numbers* $L_n = \alpha^n + \beta^n$.

Theorem 3. *For $t \equiv 2 \bmod 4$ and even d, we have*

$$X_n^{(d)} = \sum_{j=1}^{d/2} F_{2tnj+((d-1)\lambda+t)j} a_{d,j} + c_1 \cdot n + c_2,$$

where

$$a_{d,j} = \frac{1}{5^{d/2} F_{4tj}} \left\{ \begin{matrix} d \\ \frac{d}{2} - j \end{matrix} \right\}_\lambda \times \begin{cases} (-1)^{\frac{d-1}{2}-j} & \text{if } \lambda \text{ is even,} \\ (-1)^{\frac{(\frac{d-1}{2}-j)(\frac{d+1}{2}-j)}{2}} & \text{if } \lambda \text{ is odd.} \end{cases}$$

Theorem 4. *For $t \equiv 2 \bmod 4$ and odd d, we have*

$$X_n^{(d)} = \sum_{j=0}^{(d-1)/2} F_{tn(2j+1)+((d-1)\lambda+t)\frac{2j+1}{2}} a_{d,j} + c_1 \cdot n + c_2,$$

where

$$a_{d,j} = \frac{1}{5^{(d-1)/2}} L_{\frac{t}{2}(2j+1)} \left\{ \begin{matrix} d \\ \frac{d-1}{2} - j \end{matrix} \right\}_\lambda \times \begin{cases} (-1)^{\frac{d-1}{2}-j} & \text{if } \lambda \text{ is even,} \\ (-1)^{\frac{(\frac{d-1}{2}-j)(\frac{d+1}{2}-j)}{2}} & \text{if } \lambda \text{ is odd.} \end{cases}$$

3. PROOFS

For fixed t, d, λ, a computer proves the desired identity readily. And, in the first place, that is the way the identities were found. However, for general parameters, we must resort to traditional methods.

We write the product

$$\prod_{j=0}^{d-1} F_{kt+\lambda j}$$

in q-notation:

$$\mathbf{i}^{d(kt-1)+\frac{\lambda d(d-1)}{2}} q^{-\frac{d(kt-1)}{2}-\frac{\lambda d(d-1)}{4}} \frac{(q^{tk};q^{\lambda})_d}{(1-q)^d}.$$

Here, we set $q=\beta/\alpha$, $\mathbf{i}^2=-1$, and we use the notation $(x;p)_n=(1-x)(1-xp)\dots(1-xp^{n-1})$. We will also use

$$\begin{bmatrix} N \\ k \end{bmatrix}_\lambda = \frac{(q^\lambda;q^\lambda)_N}{(q^\lambda;q^\lambda)_k (q^\lambda;q^\lambda)_{N-k}}.$$

By the q-binomial theorem (or Rothe's identity, see [1]) we have

$$(q^{tk};q^\lambda)_d = \sum_{j=0}^{d} \begin{bmatrix} d \\ j \end{bmatrix}_\lambda (-1)^j q^{\lambda\binom{j}{2}} q^{tkj}$$

and so the product takes the form

$$\prod_{j=0}^{d-1} F_{kt+\lambda j} = \frac{\mathbf{i}^{d(kt-1)+\frac{\lambda d(d-1)}{2}} q^{-\frac{d(kt-1)}{2}-\frac{\lambda d(d-1)}{4}}}{(1-q)^d} \sum_{j=0}^{d} \begin{bmatrix} d \\ j \end{bmatrix}_\lambda (-1)^j q^{\lambda\binom{j}{2}} q^{tkj}.$$

Now we again convert it back into Fibonacci-type form:

$$\prod_{j=0}^{d-1} F_{kt+\lambda j} = \frac{\alpha^{dkt+\frac{\lambda d(d-1)}{2}}}{5^{d/2}} \sum_{j=0}^{d} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^j (-1)^{\lambda\frac{j(j-1)}{2}} \alpha^{j(\lambda-d\lambda-2kt)}.$$

First, we consider the case d is even to prove Theorems 1 and 3.

We reorganize the sum on j:

$$\begin{aligned}
\prod_{j=0}^{d-1} F_{kt+\lambda j} &= \frac{\alpha^{dkt+\frac{\lambda d(d-1)}{2}}}{5^{d/2}} \sum_{j=0}^{d/2-1} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{j(\lambda-d\lambda-2kt)} \\
&\quad + \frac{\alpha^{dkt+\frac{\lambda d(d-1)}{2}}}{5^{d/2}} \sum_{j=d/2+1}^{d} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{j(\lambda-d\lambda-2kt)} + \mathit{constant} \\
&= \frac{\alpha^{dkt+\frac{\lambda d(d-1)}{2}}}{5^{d/2}} \sum_{j=0}^{d/2-1} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{j(\lambda-d\lambda-2kt)} \\
&\quad + \frac{\alpha^{dkt+\frac{\lambda d(d-1)}{2}}}{5^{d/2}} \sum_{j=0}^{d/2-1} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{(d-j)(\lambda(d-j-1)+2)}{2}} \alpha^{(d-j)(\lambda-d\lambda-2kt)} + \mathit{constant}
\end{aligned}$$

$$= \frac{1}{5^{d/2}} \sum_{j=0}^{d/2-1} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)}$$

$$+ \frac{1}{5^{d/2}} \sum_{j=0}^{d/2-1} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \beta^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)} + constant$$

$$= \frac{1}{5^{d/2}} \sum_{j=0}^{d/2-1} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\lambda\frac{j(j-1)}{2}+j} L_{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)} + constant$$

$$= \frac{1}{5^{d/2}} \sum_{j=1}^{d/2} \begin{Bmatrix} d \\ \frac{d}{2}-j \end{Bmatrix}_\lambda (-1)^{\lambda\frac{(\frac{d}{2}-j)(\frac{d}{2}-j-1)}{2}+\frac{d}{2}-j} L_{2ktj+\lambda j(d-1)} + constant.$$

Now we have the formula

$$\sum_{k=0}^{n} L_{4ak+b} = \frac{F_{4an+2a+b}}{F_{2a}} + constant, \tag{1}$$

which is easy to derive using the Binet formula.

Therefore we get by summing on k

$$\sum_{k=0}^{n} \prod_{j=0}^{d-1} F_{kt+\lambda j} = \frac{1}{5^{d/2}} \sum_{j=1}^{d/2} \begin{Bmatrix} d \\ \frac{d}{2}-j \end{Bmatrix}_\lambda (-1)^{\lambda\frac{(\frac{d}{2}-j)(\frac{d}{2}-j-1)}{2}+\frac{d}{2}-j} \frac{F_{2tnj+((d-1)\lambda+t)j}}{F_{tj}}$$
$$+ constant \cdot n + constant.$$

A quick check tells us that this is the desired formula for both parities of λ.

This proof works for Theorem 3 as well, since we only needed that t was even.

In order to prove Theorems 2 and 4, we now assume that d is odd.

$$\prod_{j=0}^{d-1} F_{kt+\lambda j} = \frac{1}{5^{d/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)}$$

$$+ \frac{1}{5^{d/2}} \sum_{j=(d+1)/2}^{d} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)}$$

$$= \frac{1}{5^{d/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)}$$

$$+ \frac{1}{5^{d/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{(d-j)(\lambda(d-j-1)+2)}{2}}$$

$$
\begin{aligned}
&\times (-1)^{dkt+\frac{\lambda d(d-1)}{2}+(d-j)(\lambda-d\lambda-2kt)}\beta^{-dkt-\frac{\lambda d(d-1)}{2}-(d-j)(\lambda-d\lambda-2kt)}\\
&= \frac{1}{5^{d/2}}\sum_{j=0}^{(d-1)/2}\begin{Bmatrix} d\\ j\end{Bmatrix}_\lambda (-1)^{\lambda\frac{j(j-1)}{2}+j}\alpha^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)}\\
&\quad - \frac{1}{5^{d/2}}\sum_{j=0}^{(d-1)/2}\begin{Bmatrix} d\\ j\end{Bmatrix}_\lambda (-1)^{\lambda\frac{j(j-1)}{2}+j}\beta^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)}\\
&= \frac{1}{5^{(d-1)/2}}\sum_{j=0}^{(d-1)/2}\begin{Bmatrix} d\\ j\end{Bmatrix}_\lambda (-1)^{\lambda\frac{j(j-1)}{2}+j}F_{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)}\\
&= \frac{1}{5^{(d-1)/2}}\sum_{j=0}^{(d-1)/2}\begin{Bmatrix} d\\ \frac{d-1}{2}-j\end{Bmatrix}_\lambda (-1)^{\lambda\frac{(\frac{d-1}{2}-j)(\frac{d-1}{2}-j-1)}{2}+\frac{d-1}{2}-j}\\
&\quad\times F_{(2j+1)kt+\lambda(d-1)\frac{2j+1}{2}}.
\end{aligned}
$$

We have

$$
\sum_{k=0}^{n} F_{4ak+b} = \frac{1}{5}\frac{L_{4an+2a+b}}{F_{2a}} + constant
$$

and therefore for $t \equiv 0 \bmod 4$

$$
\begin{aligned}
\sum_{k=0}^{n}\prod_{j=0}^{d-1} F_{kt+\lambda j} &= \frac{1}{5^{(d-1)/2}}\sum_{j=0}^{(d-1)/2}\begin{Bmatrix} d\\ \frac{d-1}{2}-j\end{Bmatrix}_\lambda\\
&\quad\times (-1)^{\lambda\frac{(\frac{d-1}{2}-j)(\frac{d-1}{2}-j-1)}{2}+\frac{d-1}{2}-j}\sum_{k=0}^{n} F_{(2j+1)kt+\lambda(d-1)\frac{2j+1}{2}}\\
&= \frac{1}{5^{(d+1)/2}}\sum_{j=0}^{(d-1)/2}\begin{Bmatrix} d\\ \frac{d-1}{2}-j\end{Bmatrix}_\lambda (-1)^{\lambda\frac{(\frac{d-1}{2}-j)(\frac{d-1}{2}-j-1)}{2}+\frac{d-1}{2}-j}\\
&\quad\times \frac{L_{(2j+1)tn+(2j+1)\frac{t}{2}+\lambda(d-1)\frac{2j+1}{2}}}{F_{(2j+1)\frac{t}{2}}} + constant,
\end{aligned}
$$

which settles Theorem 2.

We have

$$
\sum_{k=0}^{n} F_{2(2a+1)k+b} = \frac{F_{2(2a+1)n+2a+1+b}}{L_{2a+1}} + constant
$$

and therefore for $t \equiv 2 \bmod 4$

$$\sum_{k=0}^{n}\prod_{j=0}^{d-1}F_{kt+\lambda j}=\frac{1}{5^{(d-1)/2}}\sum_{j=0}^{(d-1)/2}\left\{\begin{matrix} d \\ \frac{d-1}{2}-j\end{matrix}\right\}_{\lambda}(-1)^{\lambda\frac{(\frac{d-1}{2}-j)(\frac{d-1}{2}-j-1)}{2}+\frac{d-1}{2}-j}$$
$$\times\frac{F_{(2j+1)nt+(\lambda(d-1)+t)\frac{2j+1}{2}}}{L_{(2j+1)\frac{t}{2}}}+\mathit{constant},$$

which settles Theorem 4.

Thus all formulæ have been proved.

4. Conclusion

The four theorems we gave are in terms of Fibonacci numbers. Of course, with some patience, all this can be done *mutatis mutandis* for Lucas numbers and similar expressions.

Our use of computers was of the type “human–machine interaction”. It would be a good students project to provide some fully automatic programs related to our search. As general references we would like to mention [9, 6].

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