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Two Variants of the Reciprocal Super Catalan Matrix

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Abstract. In this paper, we define two kinds variants of the super Catalan matrix as well as their q-analoques. We give explicit expressions for LU-decompositions of these matrices and their inverses.

1. Introduction

For a given sequence $\{a_n\}_{n=0}^{\infty}$, the Hankel matrix is defined by

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots \\ a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & a_4 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

One can obtain a combinatorial matrix having interesting properties from a Hankel matrix. For example, the Hilbert matrix $H_n = [h_{ij}]$ is defined by $h_{ij} = \frac{1}{i+j-1}$ (for more details, see [2, 4]) and the Filbert matrix $\mathcal{F}_n = [f_{ij}]$ is defined by

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$f_{ij} = \frac{1}{F_{i+j-1}}$ (see [1, 6]). Clearly they are of the forms

$$H_n = \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \text{ and } \mathcal{F}_n = \begin{pmatrix} \frac{1}{F_0} & \frac{1}{F_1} & \frac{1}{F_2} & \cdots \\ \frac{1}{F_1} & \frac{1}{F_2} & \frac{1}{F_3} & \cdots \\ \frac{1}{F_2} & \frac{1}{F_3} & \frac{1}{F4} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where F_n is nth Fibonacci number.

Throughout this paper, we will use the q-Pochhammer symbol

$$(x;q)_n = (1-x)(1-xq)\ldots\left(1-xq^{n-1}\right)$$

and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

It is clearly that

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}, \tag{1.1}$$

where $\binom{n}{k}$ is the usual binomial coefficient.

The Cauchy binomial theorem is given by

$$\sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=1}^{n} \left(1+xq^k\right),$$

and Rothe's formula (see [2]) is given by

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x;q)_n = \prod_{k=0}^{n-1} \left(1-xq^k\right).$$

Prodinger [3] consider the reciprocal super Catalan matrix M with the entries $m_{ij} = \frac{i!(i+j)!j!}{(2i)!(2j)!}$ and obtain explicit formulae for its LU-decomposition, the LU-decomposition of its inverse, and some related matrices. For all results, q-analogues are also presented.

We rewrite the matrix M in terms of three binomial coefficients, two of them is in the reciprocal form, as shown

$$m_{ij} = \frac{i!(i+j)!j!}{(2i)!(2j)!} = \frac{i!j!}{(2i)!(2j)!} = \binom{2i}{i}^{-1} \binom{2j}{j}^{-1} \binom{i+j}{i}.$$

By inspiring the matrix M, we consider two kinds variants of it by keeping only one binomial coefficient in reciprocal form in the first one and two binomial coefficients in reciprocal forms in the second one. Also we will add two additional parameters to each one. Now we define these matrices: The first one is the matrix $A = [a_{ij}]$ of order n with the entries

$$a_{ij} = \binom{2i+m}{i}\binom{2j+t}{j}^{-1}\binom{i+j}{i}$$

and the second one is the $n \times n$ matrix B with the entries

$$b_{ij} = \binom{2i+m}{i}^{-1}\binom{2j+t}{j}\binom{i+j}{i}^{-1},$$

where m and t are nonnegative integers and all indices of these matrices start at $(0,0)$.

We write the matrices $\mathcal{A}$ and $\mathcal{B}$ which are the q-analagues of the matrices A and B, respectively. We give explicit expressions for LU-decompositions of these matrices and their inverses.

By help of a computer, LU-decompositions of these matrices were firstly obtained and then we have achieved the formulas by certain skills especially guessing skill. Using q-Zeilberger algorithm [5] and elementary matrix operations, the proofs are given as combinatorial identities. We will discuss a few of them here rather than all of them.

2. Decomposition of the Matrix $\mathcal{A}$

The matrix $\mathcal{A} = [\hat{a}_{ij}]$ has the entries $\hat{a}_{ij} = \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q \begin{bmatrix} 2j+t \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} i+j \\ i \end{bmatrix}_q$ for $0 \le i,j \le n-1$. Now we will give expressions for LU-decompositions $L_1U_1 = A$ and $L_2U_2 = A^{-1}$, for L_1^{-1}, U_1^{-1} and for L_2^{-1}, U_2^{-1} by the following theorem.

Theorem 2.1. *For m,$t \ge 0$,*

$$\begin{aligned}
(L_1)_{ij} &= \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} 2j+m \\ j \end{bmatrix}_q^{-1}, \\
\left(L_1^{-1}\right)_{ij} &= (-1)^{i-j} q^{\binom{i-j}{2}} \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} 2j+m \\ j \end{bmatrix}_q^{-1}, \\
(U_1)_{ij} &= q^{i^2-1} \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} 2j+t \\ j \end{bmatrix}_q^{-1}, \\
\left(U_1^{-1}\right)_{ij} &= (-1)^{i-j} q^{\binom{j-i}{2}-j^2+1} \begin{bmatrix} 2i+t \\ i \end{bmatrix}_q \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} 2j+m \\ j \end{bmatrix}_q^{-1},
\end{aligned}$$

$$
\begin{aligned}
(L_2)_{ij} &= (-1)^{i-j}\, q^{\binom{n-i-1}{2}-\binom{n-j-1}{2}} \begin{bmatrix} 2i+t \\ i \end{bmatrix}_q \begin{bmatrix} n-j-1 \\ i-j \end{bmatrix}_q \begin{bmatrix} 2j+1 \\ j \end{bmatrix}_q \\
&\quad \times \begin{bmatrix} i+j+1 \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+t \\ j \end{bmatrix}_q^{-1}, \\
\left(L_2^{-1}\right)_{ij} &= q^{(n-i-1)(j-i)} \begin{bmatrix} i+j \\ j \end{bmatrix}_q \begin{bmatrix} n-j-1 \\ i-j \end{bmatrix}_q \begin{bmatrix} 2i+t \\ i \end{bmatrix}_q \begin{bmatrix} 2i \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+t \\ j \end{bmatrix}_q^{-1}, \\
(U_2)_{ij} &= (-1)^{i-j}\, q^{\binom{n-j-1}{2}-\binom{n-i-1}{2}-(n-i-1)(2i+1)} \begin{bmatrix} 2i+t \\ i \end{bmatrix}_q \begin{bmatrix} n+i \\ i+j+1 \end{bmatrix}_q, \\
&\quad \times \begin{bmatrix} i+j \\ j-i \end{bmatrix}_q \begin{bmatrix} i+j \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+m \\ j \end{bmatrix}_q^{-1}, \\
\left(U_2^{-1}\right)_{ij} &= q^{(n-j-1)(i+j+1)} \begin{bmatrix} 2j+1 \\ j-i \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_q \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q \begin{bmatrix} 2j+t \\ j \end{bmatrix}_q^{-1} \\
&\quad \times \begin{bmatrix} n+j \\ i+j+1 \end{bmatrix}_q^{-1},
\end{aligned}
$$

and

$$
\det A = \prod_{k=0}^{n-1} q^{k^2} \begin{bmatrix} 2k+m \\ k \end{bmatrix}_q \begin{bmatrix} 2k+t \\ k \end{bmatrix}_q^{-1}.
$$

Proof. To prove $L_1 L_1^{-1} = I_n$ where I_n is the identity matrix of order n, consider

$$
\sum_{j \le k \le i} (L_1)_{ik} \left(L_1^{-1}\right)_{kj} = (-1)^j \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q \begin{bmatrix} 2j+m \\ j \end{bmatrix}_q^{-1} \sum_{j \le k \le i} (-1)^k q^{\binom{k-j}{2}} \begin{bmatrix} i \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q .
$$

Since

$$
\begin{bmatrix} i \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q = \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} i-j \\ k-j \end{bmatrix}_q ,
$$

we have

$$
\begin{aligned}
&\sum_{j \le k \le i} (L_1)_{ik} \left(L_1^{-1}\right)_{kj} \\
&= (-1)^j \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q \begin{bmatrix} 2j+m \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} i \\ j \end{bmatrix}_q \sum_{j \le k \le i} (-1)^k q^{\binom{k-j}{2}} \begin{bmatrix} i-j \\ k-j \end{bmatrix}_q \\
&= \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q \begin{bmatrix} 2j+m \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} i \\ j \end{bmatrix}_q \sum_{0 \le k \le i-j} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} i-j \\ k \end{bmatrix}_q .
\end{aligned}
$$

Using Rothe's formula, we see that $(1;q)_{i-j}$ is equal to 1 if $i=j$ and 0 otherwise. Then we get

$$\sum_{j\le k\le i}(L_1)_{ik}\left(L_1^{-1}\right)_{kj}=\delta_{i,j},$$

as claimed, where $\delta_{i,j}$ is Kronecker delta. For U_1 and U_1^{-1}, we write

$$\begin{aligned}
&\sum_{i\le k\le j}(U_1)_{ik}\left(U_1^{-1}\right)_{kj}\\
&=q^{i^2}\begin{bmatrix}2i+m\\ i\end{bmatrix}_q\begin{bmatrix}2j+m\\ j\end{bmatrix}_q^{-1}\sum_{i\le k\le j}(-1)^{k-j}q^{\binom{j-k}{2}-j^2}\begin{bmatrix}k\\ i\end{bmatrix}_q\begin{bmatrix}j\\ k\end{bmatrix}_q\\
&=(-1)^j\,q^{i^2-j^2}\begin{bmatrix}2i+m\\ i\end{bmatrix}_q\begin{bmatrix}2j+m\\ j\end{bmatrix}_q^{-1}\begin{bmatrix}j\\ i\end{bmatrix}_q\sum_{i\le k\le j}(-1)^kq^{\binom{j-k}{2}}\begin{bmatrix}j-i\\ k-i\end{bmatrix}_q\\
&=(-1)^{i+j}\,q^{i^2-j^2+\binom{j}{2}+\binom{i+1}{2}-ij}\begin{bmatrix}2i+m\\ i\end{bmatrix}_q\begin{bmatrix}2j+m\\ j\end{bmatrix}_q^{-1}\begin{bmatrix}j\\ i\end{bmatrix}_q\\
&\times\sum_{0\le k\le j-i}q^{\binom{k+1}{2}}\begin{bmatrix}j-i\\ k\end{bmatrix}_q\left(-q^{i-j}\right)^k.
\end{aligned}$$

By the Cauchy binomial theorem, for $i\neq j$, we get

$$\sum_{0\le k\le j-i}q^{\binom{k+1}{2}}\begin{bmatrix}j-i\\ k\end{bmatrix}_q\left(-q^{i-j}\right)^k=\prod_{k=1}^{j-i}(1-q^{i-j+k})=0.$$

Then

$$\sum_{i\le k\le j}(U_1)_{ik}\left(U_1^{-1}\right)_{kj}=\delta_{i,j}.$$

For LU-decomposition, we will show

$$\sum_{0\le k\le\min\{i,j\}}(L_1)_{ik}\,(U_1)_{kj}=\hat{a}_{ij},$$

where $A=[\hat{a}_{ij}]$. Then

$$\begin{aligned}
&\sum_{0\le k\le\min\{i,j\}}(L_1)_{ik}\,(U_1)_{kj}\\
&=\begin{bmatrix}2i+m\\ i\end{bmatrix}_q\begin{bmatrix}2j+t\\ j\end{bmatrix}_q^{-1}\sum_{0\le k\le\min\{i,j\}}q^{k^2-1}\begin{bmatrix}i\\ k\end{bmatrix}_q\begin{bmatrix}j\\ k\end{bmatrix}_q\\
&=q^{-1}\begin{bmatrix}2i+m\\ i\end{bmatrix}_q\begin{bmatrix}2j+t\\ j\end{bmatrix}_q^{-1}(q;q)_i\,(q;q)_j\\
&\times\sum_{0\le k\le\min\{i,j\}}q^{k^2}\frac{1}{(q;q)_{i-k}\,(q;q)_{j-k}\,(q;q)_k\,(q;q)_k}.
\end{aligned}$$

Denote the last sum in the above equation by $\mathtt{SUM}_k$. The Mathematica version of the q-Zeilberger algorithm [5] produces the recursion

$$\mathtt{SUM}_i = \frac{(1-q^{i+j})}{(1-q^i)^2}\mathtt{SUM}_{i-1}.$$

Since $\mathtt{SUM}_0 = \frac{1}{(q;q)_i (q;q)_j}$, we obtain

$$\mathtt{SUM}_i = \begin{bmatrix} i+j \\ i \end{bmatrix}_q \frac{1}{(q;q)_i (q;q)_j}.$$

Then we write

$$\sum_{0\le k\le \min\{i,j\}} (L_1)_{ik} (U_1)_{kj} = \hat{a}_{ij}.$$

For L_2 and L_2^{-1}, we have

$$\begin{aligned}
&\sum_{j\le k\le i} (L_2)_{ik} \left(L_2^{-1}\right)_{kj} \\
&= (-1)^i q^{\binom{n-i-1}{2}} \begin{bmatrix} 2i+t \\ i \end{bmatrix}_q \begin{bmatrix} 2j+t \\ j \end{bmatrix}_q^{-1} \\
&\times \sum_{j\le k\le i} (-1)^k q^{-\binom{n-k-1}{2}+(n-k-1)(j-k)} \begin{bmatrix} n-k-1 \\ i-k \end{bmatrix}_q \begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q \\
&\times \begin{bmatrix} i+k+1 \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} k+j \\ j \end{bmatrix}_q \begin{bmatrix} n-j-1 \\ k-j \end{bmatrix}_q \begin{bmatrix} 2k \\ k \end{bmatrix}_q^{-1} \\
&= (-1)^i q^{(n-1)(j+1)-\binom{n}{2}} \begin{bmatrix} 2i+t \\ i \end{bmatrix}_q \begin{bmatrix} 2j+t \\ j \end{bmatrix}_q^{-1} \frac{(q\,;q)_i (q\,;q)_{n-j-1}}{(q\,;q)_j (q\,;q)_{n-i-1}} \\
&\times \sum_{j\le k\le i} (-1)^k q^{\binom{k}{2}-kj} \frac{(q\,;q)_{2k+1} (q\,;q)_{k+j}}{(q\,;q)_{i-k} (q\,;q)_{i+k+1} (q\,;q)_{k-j} (q\,;q)_{2k}}.
\end{aligned}$$

By the q-Zeilberger algorithm for the second sum in the last equation, we obtain that it is equal to 0 provided that $i \ne j$. If $i = j$, it is obvious that $(L_2)_{ik} \left(L_2^{-1}\right)_{kj} = 1$. Thus

$$\sum_{j\le k\le i} (L_2)_{ik} \left(L_2^{-1}\right)_{kj} = \delta_{i,j},$$

as claimed. Similarly we have

$$\sum_{i\le k\le j} (U_2)_{ik} \left(U_2^{-1}\right)_{kj} = \delta_{i,j}.$$

For the $LU-$decomposition of $\mathcal{A}^{-1}$,we should that $\mathcal{A}^{-1} = L_2U_2$ which is same as $\mathcal{A} = U_2^{-1}L_2^{-1}$. So it is sufficient to show that

$$\sum_{\max\{i,j\}\leq k\leq n-1} \left(U_2^{-1}\right)_{ik} \left(L_2^{-1}\right)_{kj} = \hat{a}_{ij}.$$

Hence

$$\begin{aligned}
&\sum_{\max\{i,j\}\leq k\leq n-1} \left(U_2^{-1}\right)_{ik} \left(L_2^{-1}\right)_{kj} = \begin{bmatrix}2i+m\\ i\end{bmatrix}_q \begin{bmatrix}2j+t\\ j\end{bmatrix}_q^{-1} \\
&\times \sum_{\max\{i,j\}\leq k\leq n-1} q^{(n-k-1)(i+k+1)+(n-k-1)(j-k)} \begin{bmatrix}2k+1\\ k-i\end{bmatrix}_q \begin{bmatrix}i+k\\ i\end{bmatrix}_q \\
&\times \begin{bmatrix}n+k\\ i+k+1\end{bmatrix}_q^{-1} \begin{bmatrix}k+j\\ j\end{bmatrix}_q \begin{bmatrix}n-j-1\\ k-j\end{bmatrix}_q \begin{bmatrix}2k\\ k\end{bmatrix}_q^{-1} \\
&= q^{n(i+j-1)} \begin{bmatrix}2i+m\\ i\end{bmatrix}_q \begin{bmatrix}2j+t\\ j\end{bmatrix}_q^{-1} \\
&\times \sum_{\max\{i,j\}\leq k\leq n-1} q^{(n-k-1)(i+j+1)} \begin{bmatrix}2k+1\\ k-i\end{bmatrix}_q \begin{bmatrix}i+k\\ i\end{bmatrix}_q \\
&\times \begin{bmatrix}k+j\\ j\end{bmatrix}_q \begin{bmatrix}n-j-1\\ k-j\end{bmatrix}_q \begin{bmatrix}2k\\ k\end{bmatrix}_q^{-1} \begin{bmatrix}n+k\\ i+k+1\end{bmatrix}_q^{-1}.
\end{aligned}$$

If we take $(n+1)$ instead of n, we write

(2.1)
$$\sum_{j\leq k\leq n} q^{(n-k)(i+j+1)} \begin{bmatrix}2k+1\\ k-i\end{bmatrix}_q \begin{bmatrix}i+k\\ i\end{bmatrix}_q \begin{bmatrix}k+j\\ j\end{bmatrix}_q \begin{bmatrix}n-j\\ k-j\end{bmatrix}_q \begin{bmatrix}2k\\ k\end{bmatrix}_q^{-1} \begin{bmatrix}n+k+1\\ i+k+1\end{bmatrix}_q^{-1}.$$

Denote sum in (2.1) by $\texttt{SUM}_n$. For $i \neq n$ and $j \neq n$, the q-Zeilberger algorithm gives the following recursion

$$\texttt{SUM}_n = \texttt{SUM}_{n-1}.$$

Thus, $\texttt{SUM}_n = \texttt{SUM}_j = \begin{bmatrix}i+j\\ i\end{bmatrix}_q$ which completes the proof except the case $(i,j) = (n-1, n-1)$, which could be easily checked. The proof is obtained. □

3. The Decomposition of the Matrix $\mathcal{B}$

In this section, the matrix $\mathcal{B} = \left[\hat{b}_{ij}\right]$ is defined with entries

$$\hat{b}_{ij} = \begin{bmatrix}2i+m\\ i\end{bmatrix}_q^{-1} \begin{bmatrix}2j+t\\ j\end{bmatrix}_q \begin{bmatrix}i+j\\ i\end{bmatrix}_q^{-1}$$

for $0 \le i, j \le n-1$. Now we will give expressions for LU-decompositions $L_3U_3 = \mathcal{B}$ and $L_4U_4 = \mathcal{B}^{-1}$, for L_3^{-1}, U_3^{-1} and for L_4^{-1}, U_4^{-1} without proof by the following theorem

Theorem 3.1. For $m, t \ge 0$ and $0 \le i, j \le n-1$,

$$(L_3)_{ij} = \begin{bmatrix} 2j \\ j \end{bmatrix}_q \begin{bmatrix} 2j+m \\ j \end{bmatrix}_q \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} i+j \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q^{-1},$$

$$\left(L_3^{-1}\right)_{ij} = (-1)^{i-j} q^{\binom{i-j}{2}} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \begin{bmatrix} 2j+m \\ j \end{bmatrix}_q \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} 2i-1 \\ i-1 \end{bmatrix}_q^{-1} \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q^{-1},$$

$$(U_3)_{ij} = (-1)^i q^{i^2+\binom{i}{2}} \begin{bmatrix} 2j+t \\ j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ j-i \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q^{-1},$$

$$\left(U_3^{-1}\right)_{ij} = (-1)^i q^{\binom{j-i}{2}-\binom{j}{2}-j^2} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} 2j+m \\ j \end{bmatrix}_q \begin{bmatrix} 2j \\ j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q \begin{bmatrix} 2i+t \\ i \end{bmatrix}_q^{-1},$$

$$(L_4)_{ij} = (-1)^{i-j} q^{-\binom{n-j-1}{2}+\binom{n-i-1}{2}} \begin{bmatrix} n+i-1 \\ i \end{bmatrix}_q \begin{bmatrix} 2j+t \\ j \end{bmatrix}_q \begin{bmatrix} n-j+1 \\ i-j \end{bmatrix}_q$$
$$\times \begin{bmatrix} 2i+t \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} n+j-1 \\ j \end{bmatrix}_q^{-1},$$

$$\left(L_4^{-1}\right)_{ij} = q^{(n-i-1)(j-i)} \begin{bmatrix} 2j+t \\ j \end{bmatrix}_q \begin{bmatrix} n-j-1 \\ i-j \end{bmatrix}_q \begin{bmatrix} n+i-1 \\ i-j \end{bmatrix}_q \begin{bmatrix} 2i+t \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} i \\ j \end{bmatrix}_q^{-1},$$

$$(U_4)_{ij} = (-1)^{n-j-1} q^{\binom{n-j-1}{2}+(i+n-1)(i-n+1)} \begin{bmatrix} n+j-1 \\ i \end{bmatrix}_q \begin{bmatrix} n-i+j-1 \\ j \end{bmatrix}_q$$
$$\times \begin{bmatrix} n-i-1 \\ j-i \end{bmatrix}_q \begin{bmatrix} 2j+m \\ j \end{bmatrix}_q \begin{bmatrix} 2i+t \\ i \end{bmatrix}_q^{-1}.$$

$$(U_4)_{ij}^{-1} = (-1)^j q^{\binom{n}{2}-\binom{j+1}{2}+i(n-j-1)} \begin{bmatrix} 2j+t \\ j \end{bmatrix}_q \begin{bmatrix} n-i-1 \\ j-i \end{bmatrix}_q \begin{bmatrix} n+i-j-1 \\ i \end{bmatrix}_q^{-1}$$
$$\times \begin{bmatrix} n+i-1 \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2i+m \\ i \end{bmatrix}_q^{-1}$$

and

$$\det \mathcal{B} = \prod_{k=0}^{n-1} (-1)^k q^{k(3k-1)/2} \begin{bmatrix} 2k+t \\ t \end{bmatrix}_q \begin{bmatrix} 2k+m \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} k+t \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q^{-1}.$$

4. The Matrix A

In this section, we have the following results without proof by using the results of Theorem 2.1 with the fact given in (1.1). For $0 \le i, j < n-1$,

$$\begin{aligned}
(L_1)_{ij} &= \binom{2i+m}{i}\binom{i}{j}\binom{2j+m}{j}^{-1}, \\
\left(L_1^{-1}\right)_{ij} &= (-1)^{i-j}\binom{2i+m}{i}\binom{i}{j}\binom{2j+m}{j}^{-1}, \\
(U_1)_{ij} &= \binom{2i+m}{i}\binom{j}{i}\binom{2j+t}{j}^{-1}, \\
\left(U_1^{-1}\right)_{ij} &= (-1)^{i-j}\binom{2i+t}{i}\binom{j}{i}\binom{2j+m}{j}^{-1}, \\
(L_2)_{ij} &= (-1)^{i-j}\binom{2i+t}{i}\binom{n-j-1}{i-j}\binom{2j+1}{j}\binom{i+j+1}{i}^{-1}\binom{2j+t}{j}^{-1}, \\
\left(L_2^{-1}\right)_{ij} &= \binom{i+j}{j}\binom{n-j-1}{i-j}\binom{2i+t}{i}\binom{2i}{i}^{-1}\binom{2j+t}{j}^{-1}, \\
(U_2)_{ij} &= (-1)^{i-j}\binom{2i+t}{i}\binom{n+i}{i+j+1}\binom{i+j}{j-i}\binom{i+j}{j}^{-1}\binom{2j+m}{j}^{-1}, \\
\left(U_2^{-1}\right)_{ij} &= \binom{2j+1}{j-i}\binom{i+j}{i}\binom{2i+m}{i}\binom{2j+t}{j}^{-1}\binom{n+j}{i+j+1}^{-1}, \\
\det A &= \prod_{k=0}^{n-1}\binom{2k+m}{k}\binom{2k+t}{k}^{-1}.
\end{aligned}$$

5. The Matrix B

In this section, we have the following results without proof by using the results of Theorem 2.2 with the fact given in (1.1). For $0 \le i, j < n-1$,

$$(L_3)_{ij} = \binom{2j}{j}\binom{2j+m}{j}\binom{i}{j}\binom{i+j}{j}^{-1}\binom{2i+m}{i}^{-1},$$

$$\begin{aligned}
\left(L_3^{-1}\right)_{ij} &= (-1)^{i-j}\binom{i+j-1}{j}\binom{2j+m}{j}\binom{i}{j}\binom{2i-1}{i-1}^{-1}\binom{2i+m}{i}^{-1}, \\
(U_3)_{ij} &= (-1)^{i}\binom{2j+t}{j}\binom{i+j-1}{j-i}\binom{i+j}{i}^{-1}\binom{i+j-1}{j}^{-1}\binom{2i+m}{i}^{-1}, \\
\left(U_3^{-1}\right)_{ij} &= (-1)^{i-1}\binom{j}{i}\binom{2j+m}{j}\binom{2j}{j}\binom{i+j-1}{i}\binom{2i+t}{i}^{-1}, \\
(L_4)_{ij} &= (-1)^{i-j}\binom{n+i-1}{i}\binom{2j+t}{j}\binom{n-j+1}{i-j}\binom{2i+t}{i}^{-1}\binom{n+j-1}{j}^{-1}, \\
\left(L_4^{-1}\right)_{ij} &= \binom{2j+t}{j}\binom{n-j-1}{i-j}\binom{n+i-1}{i-j}\binom{2i+t}{i}^{-1}\binom{i}{j}^{-1}, \\
(U_4)_{ij} &= (-1)^{n-j}\binom{n+j-1}{i}\binom{n-i+j-1}{j}\binom{n-i-1}{j-i}\binom{2j+m}{j} \\
&\quad \times\binom{2i+t}{i}^{-1}, \\
\left(U_4^{-1}\right)_{ij} &= (-1)^{j}\binom{2j+t}{j}\binom{n-i-1}{j-i}\binom{n+i-j-1}{i}^{-1}\binom{n+i-1}{j}^{-1} \\
&\quad \times\binom{2i+m}{i}^{-1}, \\
\det B &= \prod_{k=0}^{n-1}(-1)^{k}\binom{2k+t}{k}\binom{2k+m}{k}^{-1}\binom{k+t}{k}^{-1}\binom{2k-1}{k}^{-1}.
\end{aligned}$$

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