# CLOSED FORM EVALUATION OF SUMS CONTAINING SQUARES OF FIBONOMIAL COEFFICIENTS

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Abstract. We give a systematic approach to compute certain sums of squares of Fibonomial coefficients with finite products of generalized Fibonacci and Lucas numbers as coefficients. The technique is to rewrite everything in terms of a variable $q$, and then to use generating functions and Rothe's identity from classical $q$-calculus.

## 1. Introduction

Define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$ by

$$\begin{aligned} U_n &= pU_{n-1} + U_{n-2}, \quad U_0 = 0,\ U_1 = 1, \\ V_n &= pV_{n-1} + V_{n-2}, \quad\ \ V_0 = 2,\ \ V_1 = p. \end{aligned}$$

For $n \geq k \geq 1$ and an integer $m$, define the generalized Fibonomial coefficient with indices in an arithmetic progression by

$$\left\{ {n \atop k} \right\}_{U;m} := \frac{U_m U_{2m} \dots U_{nm}}{(U_m U_{2m} \dots U_{km})(U_m U_{2m} \dots U_{(n-k)m})}$$

with $\left\{ {n \atop 0} \right\}_{U;m} = \left\{ {n \atop n} \right\}_{U;m} = 1$. When $p = m = 1$, we obtain the usual Fibonomial coefficients, denoted by $\left\{ {n \atop k} \right\}_F$. When $m = 1$, we obtain the generalized Fibonomial coefficients, denoted by $\left\{ {n \atop k} \right\}_U$.

A special case is the $n$th central generalized Fibonomial coefficient with indices in an arithmetic progression, defined as $\left\{ {2n \atop n} \right\}_{U;m}$. When $m = p = 1$, we obtain the $n$th central Fibonomial coefficient, denoted by $\left\{ {2n \atop n} \right\}_F$. Our evaluations will be in terms of such numbers.

In this paper, we present three sets of identities which are expressed in the notion of $\left\{ {n \atop k} \right\}_{U;m}$. More importantly, we describe a general methodology how to evaluate the sums occurring in them, as well as many others.

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Our approach is as follows. We use the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n(1+q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$ where $\alpha, \beta = (p \pm \sqrt{\Delta})/2$ and $\Delta = p^2 + 4$.

Throughout this paper we will use the following notations: the $q$-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and the Gaussian $q$-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

The link between the generalized Fibonomial and Gaussian $q$-binomial coefficients is

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{U;m} = \alpha^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m} \quad \text{with} \quad q = -\alpha^{-2}.$$

We recall that one version of the *Cauchy binomial theorem* is given by

$$\sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=1}^{n} (1 + xq^k),$$

and *Rothe's* formula [1] is

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x;q)_n = \prod_{k=0}^{n-1} (1 - xq^k).$$

All the identities we will derive hold for general $q$, and results about generalized Fibonacci and Lucas numbers come out as corollaries for the special choice of $q$. We will frequently denote $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_{U;1}$ by $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_{U}$.

Recently, the authors of [4, 3] computed certain Fibonomial sums with generalized Fibonacci and Lucas numbers as coefficients. For example, if $n$ and $m$ are both nonnegative integers, then

$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix} U_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \begin{Bmatrix} 2m-1 \\ 2k-1 \end{Bmatrix} U_{(4k-2)n},$$

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix} U_{2mk} = P_{n,m} \sum_{k=0}^{m} \begin{Bmatrix} 2m \\ 2k \end{Bmatrix} U_{(2n+1)2k},$$

$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix} V_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \begin{Bmatrix} 2m-1 \\ 2k-1 \end{Bmatrix} V_{(4k-2)n},$$

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix} V_{2mk} = P_{n,m} \sum_{k=0}^{m} \begin{Bmatrix} 2m \\ 2k \end{Bmatrix} V_{(2n+1)2k},$$

where

$$P_{n,m} = \begin{cases} \prod_{k=0}^{n-m} V_{2k} & \text{if } \ n \ge m, \\ \prod_{k=1}^{m-n-1} V_{2k}^{-1} & \text{if } \ n < m; \end{cases}$$

alternating analogues of these sums were also presented. In particular, if $m$ is a small number, we can think about this as the closed form evaluation of the left-hand side in terms of the finitely many terms of the right-hand side.

In this paper we investigate similar sums, but the Fibonomial coefficients appear in squared form. The approach works for Fibonacci and Lucas (-type) numbers as factors likewise. We only discuss the Fibonacci case, but add two Lucas examples at the very end of the paper.

We would like to end this introduction by pointing out a few papers that are similar in spirit than our current investigation: [2, 5, 6].

## 2. A systematic approach

We are interested to evaluate

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_U^2 U_{\lambda_1 k + r_1} \dots U_{\lambda_s k + r_s}$$

in closed form where $r_i$ and $\lambda_i \ge 1$ are integers. For that, it will be translated into $q$-notation:

$$\begin{aligned} (1-q)^{-s} \sum_{k=0}^{n} (-1)^{k(n-1)} q^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 \mathbf{i}^{(\lambda_1+\dots+\lambda_s)k + r_1 + \dots + r_s - s} \\ \times q^{\frac{s}{2} - \frac{k}{2}(\lambda_1+\dots+\lambda_s) - \frac{1}{2}(r_1+\dots+r_s)} (1 - q^{\lambda_1 k + r_1}) \cdots (1 - q^{\lambda_s k + r_s}). \end{aligned}$$

For our method to work, the factor $(-1)^k$ must appear. That means that we have two possibilities:

- $n$ is even and $\lambda_1 + \dots + \lambda_s \equiv 0 \pmod 4$
- $n$ is odd and $\lambda_1 + \dots + \lambda_s \equiv 2 \pmod 4$

On the other hand, if

- $n$ is even and $\lambda_1 + \dots + \lambda_s \equiv 2 \pmod 4$
- $n$ is odd and $\lambda_1 + \dots + \lambda_s \equiv 0 \pmod 4$,

then we are able to evaluate

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_U^2 U_{\lambda_1 k + r_1} \dots U_{\lambda_s k + r_s} (-1)^k$$

in closed form.

Here is how it goes: Expanding the product

$$(1-q^{\lambda_1 k+r_1})\cdots(1-q^{\lambda_s k+r_s})$$

and ignoring constant factors, we have to evaluate a *finite number* of terms of the form

$$\sum_{k=0}^{n}(-1)^k q^{k^2+\mu k-nk}\begin{bmatrix}n\\k\end{bmatrix}_q^2$$

where $\mu$ is an integer. Now we will explain how this can be done.

$$\sum_{k=0}^{n}(-1)^k q^{k^2+\mu k-nk}\begin{bmatrix}n\\k\end{bmatrix}_q^2=q^{-\binom{n}{2}}\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}+\binom{n-k}{2}+\mu k}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n\\n-k\end{bmatrix}_q$$

and

$$\begin{aligned}S&=\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}+\binom{n-k}{2}+\mu k}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n\\n-k\end{bmatrix}_q\\&=[z^n]\Big(\sum_{k\ge0}(-1)^k q^{\binom{k}{2}+\mu k}\begin{bmatrix}n\\k\end{bmatrix}_q z^k\Big)\cdot\Big(\sum_{k\ge0}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q z^k\Big)\\&=[z^n](zq^{\mu};q)_n(-z;q)_n.\end{aligned}$$

The point is now that there are factors $(1-zq^i)$ and $(1+zq^i)$ that can be combined to $(1-z^2q^{2i})$. (That is the reason that we need the factor $(-1)^k$ in our sums, as mentioned before.) In fact, there are $n-|\mu|$ such pairs, and only $2|\mu|$ separate factors. They mess up the final result, but since $\mu$ is a constant (not depending on $n$), there is no principal difficulty involved. We have again to evaluate a *finite number* of terms of the form

$$[z^n]z^a q^b(z^2q^c;q^2)_{n-|\mu|}=[z^{n-a}]q^b(z^2q^c;q^2)_{n-|\mu|}.$$

This is either 0 for $n-a$ odd or

$$q^{b+\frac{c(n-a)}{2}}\begin{bmatrix}n-|\mu|\\\frac{n-a}{2}\end{bmatrix}_{q^2}(-1)^{\frac{n-a}{2}}q^{n-a}$$

otherwise.

Eventually we end up with a (finite) linear combination of terms of the form

$$\begin{bmatrix}n-|\mu|\\\frac{n-a}{2}\end{bmatrix}_{q^2}$$

for some integers $\mu$ and $a$. The final step is to translate such a result back to expressions in terms of $\left\{{n-|\mu| \atop (n-a)/2}\right\}_{U;2}$ and simplify according to the Binet formula and the recursion of second order for $U_n$.

In the remaining sections, this general program will be demonstrated in more detail on two examples, one for $n$ even and one for $n$ odd. Further, we will list several attractive formulæ that were obtained using the procedure just described.

## 3. Illustrative Examples

Now we work out four examples that fall into the general scheme mentioned above in more detail. Also we will present some further examples without proof.

**Theorem 1.** *For nonnegative $n$,*

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}^2 U_{2k}^2 = \Delta \left\{ {2n \atop n} \right\}_{U;2} \frac{U_{2n}^3 U_{2n+1}}{V_{2n-1} V_{2n}},$$

*where $\Delta$ is defined as before.*

*Proof.* First we convert the left-hand side of the claim in $q$-notation:

$$\begin{aligned}\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}^2 U_{2k}^2 &= \frac{1}{(1-q)^2} \sum_{k=0}^{2n} {2n \brack k}_q^2 \alpha^{2k(2n-k)} \alpha^{2(2k-1)} (1-q^{2k})^2 \\ &= \frac{\alpha^{-2}}{(1-q)^2} \sum_{k=0}^{2n} {2n \brack k}_q^2 \alpha^{4k+4kn-2k^2} (1-q^{2k})^2 \\ &= \frac{\alpha^{-2}}{(1-q)^2} \sum_{k=0}^{2n} {2n \brack k}_q^2 \mathbf{i}^{4k+4kn-2k^2} q^{-\frac{1}{2}(4k+4kn-2k^2)} \\ &\quad \times (1-2q^{2k}+q^{4k}) \\ &= -\frac{q}{(1-q)^2} \sum_{k=0}^{2n} {2n \brack k}_q^2 (-1)^k q^{k^2-2kn-2k} (1-2q^{2k}+q^{4k}).\end{aligned}$$

Second we convert the right-hand side of the claim in $q$-notation:

$$\begin{aligned}\Delta \left\{ {2n \atop n} \right\}_{U;2} \frac{U_{2n}^3 U_{2n+1}}{V_{2n-1} V_{2n}} &= (\alpha-\beta)^2 \alpha^{2n^2} {2n \brack n}_{q^2} \frac{\alpha^{3(2n-1)} \frac{(1-q^{2n})^3}{(1-q)^3} \alpha^{2n} \frac{(1-q^{2n+1})}{(1-q)}}{\alpha^{2n-1}(1+q^{2n-1}) \alpha^{2n}(1+q^{2n})} \\ &= \alpha^{2n(n+2)} {2n \brack n}_{q^2} \frac{(1-q^{2n})^3 (1-q^{2n+1})}{(1+q^{2n-1})(1+q^{2n})(1-q)^2} \\ &= (-1)^n q^{-n(n+2)} {2n \brack n}_{q^2} \frac{(1-q^{2n})^3 (1-q^{2n+1})}{(1+q^{2n-1})(1+q^{2n})(1-q)^2}.\end{aligned}$$

Thus we need to prove that

$$\begin{aligned}&\sum_{k=0}^{2n} {2n \brack k}_q^2 (-1)^k q^{k^2-2kn-2k} (1-2q^{2k}+q^{4k}) \\ &\qquad = (-1)^{n+1} q^{-(n+1)^2} {2n \brack n}_{q^2} \frac{(1-q^{2n})^3 (1-q^{2n+1})}{(1+q^{2n-1})(1+q^{2n})}.\end{aligned}$$

Let

$$S_1=\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}_q^2(-1)^k\,q^{k^2-2kn},\quad S_2=\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-2k}$$

$$\text{and}\quad S_3=\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}_q^2(-1)^k\,q^{k^2-2kn+2k}.$$

Now we consider $S_1$:

$$\begin{aligned}
&\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn}\\
&=q^{-2n^2+n}\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}_q\begin{bmatrix}2n\\2n-k\end{bmatrix}_q(-1)^kq^{\binom{2n-k}{2}}q^{\binom{k}{2}}\\
&=q^{-2n^2+n}\left[z^{2n}\right]\sum_{k\ge0}\begin{bmatrix}2n\\k\end{bmatrix}_qq^{\binom{k}{2}}z^k\cdot\sum_{k\ge0}\begin{bmatrix}2n\\2n-k\end{bmatrix}_q(-1)^kq^{\binom{2n-k}{2}}z^k\\
&=q^{-2n^2+n}\left[z^{2n}\right]\sum_{k\ge0}\begin{bmatrix}2n\\k\end{bmatrix}_qq^{\binom{k}{2}}z^k\cdot\sum_{k\ge0}\begin{bmatrix}2n\\k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}}z^k\\
&=q^{-2n^2+n}\left[z^{2n}\right](z;q)_{2n}(-z;q)_{2n}\\
&=q^{-2n^2+n}\left[z^{2n}\right](z^2;q^2)_{2n}\\
&=q^{-2n^2+n}\begin{bmatrix}2n\\n\end{bmatrix}_{q^2}(-1)^nq^{2\binom{n}{2}}\\
&=q^{-n^2}(-1)^n\begin{bmatrix}2n\\n\end{bmatrix}_{q^2}.
\end{aligned}$$

Now we consider $S_2$:

$$\begin{aligned}
&\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}_q^2(-1)^kq^{k^2-2k(n+1)}\\
&=q^{-2n^2+n}\left[z^{2n}\right]\sum_{k\ge0}\begin{bmatrix}2n\\k\end{bmatrix}_qq^{\binom{k}{2}-2k}z^k\cdot\sum_{k\ge0}\begin{bmatrix}2n\\2n-k\end{bmatrix}_q(-1)^kq^{\binom{2n-k}{2}}z^k\\
&=q^{-2n^2+n}\left[z^{2n}\right]\sum_{k\ge0}\begin{bmatrix}2n\\k\end{bmatrix}_qq^{\binom{k}{2}-2k}z^k\cdot\sum_{k\ge0}\begin{bmatrix}2n\\k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}}z^k\\
&=q^{-2n^2+n}\left[z^{2n}\right](z/q^2;q)_{2n}(-z;q)_{2n}\\
&=q^{-2n^2+n}\left[z^{2n}\right](1-z/q^2)(1-z/q)(z;q)_{2n-2}(1+zq^{2n-1})\\
&\times(1+zq^{2n-2})(-z;q)_{2n-2}
\end{aligned}$$

$$\begin{aligned}
&= q^{-2n^2+n}\left[z^{2n}\right](z^2;q^2)_{2n-2}\Big(1 - zq^{-2}(1-q^{2n})(1+q)\\
&\quad - q^{-3}z^2(-1-q^{4n}+q^{2n+1}+q^{2n-1}+2q^{2n})\\
&\quad + z^3q^{2n-5}(1+q)(1-q^{2n}) + q^{4n-6}z^4\Big)\\
&= q^{-2n^2+n}\left[z^{2n}\right](z^2;q^2)_{2n-2}\Big(1 - q^{-3}(-1-q^{4n}+q^{2n+1}+q^{2n-1}+2q^{2n})z^2\\
&\quad + q^{4n-6}z^4\Big),\\
&= q^{-2n^2+n}\Bigg(\begin{bmatrix}2n-2\\ n\end{bmatrix}_{q^2}(-1)^nq^{2\binom{n}{2}}\\
&\quad - q^{-3}(-1-q^{4n}+q^{2n+1}+q^{2n-1}+2q^{2n})(-1)^{n-1}q^{2\binom{n-1}{2}}\begin{bmatrix}2n-2\\ n-1\end{bmatrix}_{q^2}\\
&\quad + q^{4n-6}(-1)^{n-2}q^{2\binom{n-2}{2}}\begin{bmatrix}2n-2\\ n-2\end{bmatrix}_{q^2}\Bigg)\\
&= 2\begin{bmatrix}2n-2\\ n\end{bmatrix}_{q^2}(-1)^nq^{-n^2}\\
&\quad + q^{-n^2-2n-1}(-1-q^{4n}+q^{2n+1}+q^{2n-1}+2q^{2n})(-1)^n\begin{bmatrix}2n-2\\ n-1\end{bmatrix}_{q^2}.
\end{aligned}$$

A similar computation evaluates $S_3$:

$$\begin{aligned}
S_3 &= q^{-2n^2+n}\left[z^{2n}\right](z^2q^4;q^2)_{2n-2}\Big(1 + z(1-q^{2n})(1+q) + q^{4n+2}z^4\\
&\quad - qz^2(-1+q^{2n+1}+q^{2n-1}+2q^{2n}-q^{4n}) - z^3q^{2n+1}(1+q)(1-q^{2n})\Big)\\
&= q^{-2n^2+n}\left[z^{2n}\right](z^2q^4;q^2)_{2n-2}\Big(1 + q^{4n+2}z^4\\
&\quad - qz^2(-1+q^{2n+1}+q^{2n-1}+2q^{2n}-q^{4n})\Big)\\
&= 2q^{-n^2+4n}(-1)^n\begin{bmatrix}2n-2\\ n\end{bmatrix}_{q^2}\\
&\quad + q^{-n^2+2n-1}(-1+q^{2n+1}+q^{2n-1}+2q^{2n}-q^{4n})(-1)^n\begin{bmatrix}2n-2\\ n-1\end{bmatrix}_{q^2}.
\end{aligned}$$

Thus our sum in $q$-notation is

$$\begin{aligned}
\sum_{k=0}^{2n}\begin{bmatrix}2n\\ k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-2k}(1-2q^{2k}+q^{4k}) &= S_2 - 2S_1 + S_3\\
&= q^{-(n+1)^2}(-1)^{n+1}\begin{bmatrix}2n\\ n\end{bmatrix}_{q^2}\frac{(1-q^{2n})^3(1-q^{2n+1})}{(1+q^{2n-1})(1+q^{2n})},
\end{aligned}$$

as claimed. □

**Theorem 2.** *For nonnegative integer $n$,*
$$\sum_{k=0}^{2n+1}\begin{Bmatrix}2n+1\\k\end{Bmatrix}^2U_{2k}^2(-1)^k=-2\frac{V_1U_{2n+1}U_{2n+2}}{V_{2n}}\begin{Bmatrix}2n+1\\n\end{Bmatrix}_{U;2}.$$

*Proof.* First we convert the left-hand side of the claim in $q$-notation:
$$\sum_{k=0}^{2n+1}\begin{Bmatrix}2n+1\\k\end{Bmatrix}^2U_{2k}^2(-1)^k$$
$$=-\frac{q}{(1-q)^2}\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-3k}(1-q^{2k})^2.$$

Now we convert the right-hand side of it:
$$-2\frac{V_1U_{2n+1}U_{2n+2}}{V_{2n}}\begin{Bmatrix}2n+1\\n\end{Bmatrix}_{U;2}$$
$$=2(-1)^nq^{-(n+1)^2}\frac{(1+q)(1-q^{2n+1})(1-q^{2n+1})}{(1+q^{2n})(1-q)^2}\begin{bmatrix}2n+1\\n\end{bmatrix}_{q^2}.$$

Thus we must prove that
$$\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-3k}(1-q^{2k})^2$$
$$=2(-1)^{n+1}q^{-n^2-2n-2}\frac{(1+q)(1-q^{2n+1})(1-q^{2n+1})}{(1+q^{2n})}\begin{bmatrix}2n+1\\n\end{bmatrix}_{q^2}.$$

Let
$$S_4=\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-3k},\quad S_5=\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-k},$$
$$S_6=\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn+k}.$$

Thus for $S_4$, consider
$$S_4=\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-3k}$$
$$=q^{-2n^2-n}\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\k\end{bmatrix}_q\begin{bmatrix}2n+1\\2n+1-k\end{bmatrix}_q(-1)^kq^{\binom{2n+1-k}{2}}q^{\binom{k}{2}-2k}$$

$$
\begin{aligned}
&= q^{-2n^2-n}\left[z^{2n+1}\right]\sum_{k\geq 0}\begin{bmatrix}2n+1\\k\end{bmatrix}_q q^{\binom{k}{2}-2k}z^k\cdot\sum_{k\geq 0}\begin{bmatrix}2n+1\\k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}}z^k\\
&= q^{-2n^2-n}\left[z^{2n+1}\right](z/q^2;q)_{2n+1}(-z;q)_{2n+1}\\
&= q^{-2n^2-n}\left[z^{2n+1}\right](1-z/q)(1-z/q^2)(z;q)_{2n-1}\\
&\times(1+zq^{2n-1})(1+zq^{2n})(-z;q)_{2n-1}\\
&= q^{-2n^2-n}\left[z^{2n+1}\right]\Big((z^2;q^2)_{2n-1}(1-zq^{-2}(1+q)(1-q^{2n+1})\\
&\qquad+z^2q^{-3}(1-q^{2n+2}+q^{4n+2}-q^{2n}-2q^{2n+1})\\
&\qquad+z^3q^{2n-4}(1+q)(1-q^{2n+1})+q^{4n-4}z^4\Big)\\
&= -q^{-2n^2-n-2}\left[z^{2n}\right](z^2;q^2)_{2n-1}(1+q)(1-q^{2n+1})\\
&\qquad+q^{-2n^2+n-4}\left[z^{2n-2}\right](z^2;q^2)_{2n-1}(1+q)(1-q^{2n+1})\\
&= (1+q)(1-q^{2n+1})\Bigg((-q^{-2n^2-n-2}q^{2\binom{n}{2}}\begin{bmatrix}2n-1\\n\end{bmatrix}_{q^2}(-1)^n\\
&\qquad+q^{-2n^2+n-4}q^{2\binom{n-1}{2}}\begin{bmatrix}2n-1\\n-1\end{bmatrix}_{q^2}(-1)^{n-1}\Bigg)\\
&= 2(-1)^{n-1}q^{-n^2-2n-2}(1+q)(1-q^{2n+1})\begin{bmatrix}2n-1\\n\end{bmatrix}_{q^2}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-k}\\
&\qquad= q^{-2n^2-n}\left[z^{2n+1}\right]\sum_{k\geq 0}\begin{bmatrix}2n+1\\k\end{bmatrix}_q q^{\binom{k}{2}}z^k\cdot\sum_{k\geq 0}\begin{bmatrix}2n+1\\k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}}z^k\\
&\qquad= q^{-2n^2-n}\left[z^{2n+1}\right](z;q)_{2n+1}(-z;q)_{2n+1}\\
&\qquad= q^{-2n^2-n}\left[z^{2n+1}\right](z^2;q^2)_{2n+1}=0.
\end{aligned}
$$

A similar computation gives

$$
S_6 = 2(-1)^nq^{-n^2+2n}(1+q)(1-q^{2n+1})\begin{bmatrix}2n-1\\n\end{bmatrix}_{q^2}.
$$

Thus our sum in $q$-notation is

$$
\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-3k}(1-2q^{2k}+q^{4k}) = S_4-2S_5+S_6
$$

$$= 2(-1)^{n+1}q^{-n^2-2n-2}\frac{(1+q)(1-q^{2n+1})(1-q^{2n+2})}{(1+q^{2n})}\begin{bmatrix}2n+1\\ n\end{bmatrix}_{q^2},$$

as claimed. □

Now we will present some other results without proof:

**Theorem 3.** *For any integer $r$ and nonnegative integer $n$,*

*(1)*
$$\sum_{k=0}^{2n+1}\begin{Bmatrix}2n+1\\ k\end{Bmatrix}_U^2 U_{k+r}^2 = U_{2n+1}U_{2n+1+2r}\begin{Bmatrix}2n\\ n\end{Bmatrix}_{U;2},$$

*(2)*
$$\sum_{k=0}^{2n+1}\begin{Bmatrix}2n+1\\ k\end{Bmatrix}_U^2 U_{2k+r} = U_{2n+1}V_{2n+1+r}\begin{Bmatrix}2n\\ n\end{Bmatrix}_{U;2},$$

*(3)*
$$\sum_{k=0}^{2n+1}\begin{Bmatrix}2n+1\\ k\end{Bmatrix}_U^2 U_{4k+r}(-1)^k = -2U_2U_{2n+1}V_{4n+2+r}\begin{Bmatrix}2n-1\\ n\end{Bmatrix}_{U;2}.$$

**Theorem 4.** *For any integer $r$ and nonnegative integer $n$,*

*(1)*
$$\sum_{k=0}^{2n}\begin{Bmatrix}2n\\ k\end{Bmatrix}_U^2 = \begin{Bmatrix}2n\\ n\end{Bmatrix}_{U;2},$$

*(2)*
$$\sum_{k=0}^{2n}\begin{Bmatrix}2n\\ k\end{Bmatrix}_U^2(-1)^k U_{2k+r} = 2U_{2n+r}\begin{Bmatrix}2n-1\\ n\end{Bmatrix}_{U;2},$$

*(3)*
$$\sum_{k=0}^{2n}\begin{Bmatrix}2n\\ k\end{Bmatrix}_U^2 U_k^4 = U_{2n-1}U_{2n}^2U_{2n+1}\begin{Bmatrix}2n-2\\ n-1\end{Bmatrix}_{U;2},$$

*(4)*
$$\sum_{k=0}^{2n}\begin{Bmatrix}2n\\ k\end{Bmatrix}_U^2(-1)^k U_{3k}^2 = 2\Delta U_3U_{4n}\begin{Bmatrix}2n+1\\ 3\end{Bmatrix}_U\begin{Bmatrix}2n-3\\ n-2\end{Bmatrix}_{U;2},$$

*where $\Delta = p^2+4$ is defined as before.*

**Theorem 5.** *For nonnegative integer $n$,*

*(1)*
$$\sum_{k=0}^{2n+1}\begin{Bmatrix}2n+1\\ k\end{Bmatrix}^2 V_{2k} = \Delta U_{2n+1}^2\begin{Bmatrix}2n\\ n\end{Bmatrix}_{U;2},$$

*(2)*
$$\sum_{k=0}^{2n} \left\{ \begin{matrix} 2n \\ k \end{matrix} \right\}^2 V_{2k} \left(-1\right)^k = 2V_{2n} \left\{ \begin{matrix} 2n-1 \\ n \end{matrix} \right\}_{U;2},$$
*where* $\Delta$ *is defined as before.*

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