THE GENERALIZED LILBERT MATRIX

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Abstract. We introduce a generalized Lilbert [*L*ucas-H*ilbert*] matrix. Explicit formulæ are derived for the LU-decomposition and their inverses, as well as the Cholesky decomposition. The approach is to use q-analysis and to leave the justification of the necessary identities to the q-version of Zeilberger's celebrated algorithm.

1. Introduction

The Filbert matrix $H_n = \left(\check{h}_{ij}\right)_{i,j=1}^{n}$ is defined by $\check{h}_{ij} = \frac{1}{F_{i+j-1}}$ as an analogue of the Hilbert matrix where F_n is the nth Fibonacci number. It has been defined and studied by Richardson [7].

After the Filbert matrix, several generalizations and analogues of it have been investigated and studied by Kılıç and Prodinger. For the readers convenience, we briefly summarize these generalizations:

- In [1], Kılıç and Prodinger studied the generalized Filbert Matrix $\mathcal{F}$ with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter.
- After this generalization, Prodinger [6] defined a new generalization of the generalized Filbert matrix by introducing 3 additional parameters by taking its entries as $\frac{x^i y^j}{F_{\lambda(i+j)+r}}$.
- Recently, in [2], Kılıç and Prodinger gave a further generalization of the generalized Filbert Matrix $\mathcal{F}$ by defining the matrix $\mathcal{Q}$ with entries h_{ij} as follows
$$h_{ij} = \frac{1}{F_{i+j+r}F_{i+j+r+1}\dots F_{i+j+r+k-1}},$$
where $r \geq -1$ is an integer parameter and $k \geq 0$ is an integer parameter.
- In a further paper [4], Kılıç and Prodinger introduced a new kind of generalized Filbert matrix $\mathcal{G}$ with entries g_{ij} by
$$g_{ij} = \frac{1}{F_{\lambda(i+j)+r}F_{\lambda(i+j+1)+r}\dots F_{\lambda(i+j+k-1)+r}},$$
where $r \geq -1$ and $\lambda \geq 1$ are integer parameters.

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- More recently, in [3], Kılıç and Prodinger introduced four generalizations of the Filbert matrix H_n, and defined the matrices $\mathcal{P}$, $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{Y}$ with entries

$$p_{ij} = \frac{1}{F_{\lambda i+\mu j+r}},\ k_{ij} = \frac{F_{\lambda i+\mu j+r}}{F_{\lambda i+\mu j+s}},\ \ell_{ij} = \frac{1}{L_{\lambda i+\mu j+r}} \text{ and } y_{ij} = \frac{L_{\lambda i+\mu j+r}}{L_{\lambda i+\mu j+s}},$$

respectively, where s, r, λ and μ are integer parameters such that $s \neq r$, and $r, s \geq -1$ and $\lambda, \mu \geq 1$.

In the works summarized above, the authors derived explicit formulæ for the LU-decomposition (For any square matrix A, a decomposition $A = LU$, where L is a unit lower triangular matrix and U is an upper triangular matrix, is called LU-decomposition of A) for the matrices mentioned above. Also they derived explicit formulæ their inverses.

Let $\{U_n\}$ and $\{V_n\}$ be generalized Fibonacci and Lucas sequences, respectively, whose the Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n (1 + q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

When $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = (1-\sqrt{5})/(1+\sqrt{5})$), the sequence $\{U_n\}$ is reduced to the Fibonacci sequence $\{F_n\}$ and the sequence $\{V_n\}$ is reduced to the Lucas sequence $\{L_n\}$.

When $\alpha = 1+\sqrt{2}$ (or equivalently $q = (1-\sqrt{2})/(1+\sqrt{2})$), the sequence $\{U_n\}$ is reduced to the Pell sequence $\{P_n\}$ and the sequence $\{V_n\}$ is reduced to the Pell-Lucas sequence $\{Q_n\}$.

In this paper, we define the Lilbert matrix $\mathcal{T}$ with entries t_{ij} by

$$t_{ij} = \frac{1}{L_{\lambda(i+j)+r}L_{\lambda(i+j+1)+r}\dots L_{\lambda(i+j+k-1)+r}}.$$

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ and for $z > 1$, the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(z,y)} = \frac{(q^z;q^y)_n}{(q^z;q^y)_k (q^z;q^y)_{n-k}}$$

and for the case $z = y$, we will denote the Gaussian q-binomial coefficients as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{z} = \frac{(q^z;q^z)_n}{(q^z;q^z)_k (q^z;q^z)_{n-k}}.$$

We could also allow $z \geq 1$, but might have to take limits in some rare cases.

Furthermore, we will use *generalized Fibonomial coefficients*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{U(a,b)} = \frac{U_{b(n-1)+a}U_{b(n-2)+a}\dots U_{b(n-k)+a}}{U_a U_{b+a} U_{2b+a}\dots U_{b(k-1)+a}}$$

with $\left\{ {n \atop 0} \right\}_{U(a,b)} = 1$ where U_n is the nth generalized Fibonacci number.

For $a = b$, we denote the generalized Fibonomial coefficients as $\left\{ {n \atop k} \right\}_{U(a)}$. Especially for $a = b = 1$, the generalized Fibonomial coefficients are denoted by $\left\{ {n \atop k} \right\}_{U}$. When $U_n = F_n$, the generalized Fibonomial reduces to the Fibonomial coefficients denoted by $\left\{ {n \atop k} \right\}_{F}$:

$$\left\{ {n \atop k} \right\}_{F} = \frac{F_n F_{n-1} \dots F_{n-k+1}}{F_1 F_2 \dots F_k}.$$

Similarly, when $U_n = P_n$, the generalized Fibonomial reduces to the Pellnomial coefficients denoted by $\left\{ {n \atop k} \right\}_{P}$:

$$\left\{ {n \atop k} \right\}_{P} = \frac{P_n P_{n-1} \dots P_{n-k+1}}{P_1 P_2 \dots P_k}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ {n \atop k} \right\}_{(z,y)} = \alpha^{yk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{(z,y)} \quad \text{with} \quad q = -\alpha^{-2}.$$

Furthermore, we will use *generalized Lucanomial coefficients*

$$\left\langle {n \atop k} \right\rangle_{V(a,b)} = \frac{V_{b(n-1)+a} V_{b(n-2)+a} \dots V_{b(n-k)+a}}{V_a V_{b+a} V_{2b+a} \dots V_{b(k-1)+a}}$$

with $\left\langle {n \atop 0} \right\rangle_{(a,b)} = 1$ where V_n is the nth generalized Lucas number.

For $a = b$, we denote the generalized Lucanomial coefficients as $\left\langle {n \atop k} \right\rangle_{V(a)}$. Especially for $a = b = 1$, the generalized Lucanomial coefficients are denoted by $\left\langle {n \atop k} \right\rangle_{V}$. When $V_n = L_n$, the generalized Lucanomial coefficients are reduced to the Lucanomial coefficients denoted by $\left\langle {n \atop k} \right\rangle_{L}$:

$$\left\langle {n \atop k} \right\rangle_{L} = \frac{L_n L_{n-1} \dots L_{n-k+1}}{L_1 L_2 \dots L_k}.$$

When $V_n = Q_n$, the generalized Lucanomial coefficients are reduced to the Pell-Lucanomial coefficients denoted by $\left\langle {n \atop k} \right\rangle_{Q}$:

$$\left\langle {n \atop k} \right\rangle_{Q} = \frac{Q_n Q_{n-1} \dots Q_{n-k+1}}{Q_1 Q_2 \dots Q_k}.$$

The link between the generalized Lucanomial and Gaussian q-binomial coefficients is

$$\left\langle {n \atop k} \right\rangle_{V(z,y)} = \alpha^{yk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{(-z,y)} \quad \text{with} \quad q = -\alpha^{-2}.$$

Considering the definitions of the matrix $\mathcal{T}$ and the q-Pochhammer symbol, we rewrite the matrix $\mathcal{T} = [t_{ij}]$ as

$$t_{ij} = \mathbf{i}^{k(\lambda(i+j)+r)+\frac{\lambda k(k-1)}{2}} q^{-\frac{k}{2}(\lambda(i+j)+r)-\frac{\lambda k(k-1)}{4}} \left(-q^{\lambda(i+j)+r}; q^{\lambda}\right)_k .$$

We call the matrix $\mathcal{T}_n$ the *generalized Lilbert matrix.*

We will derive explicit formulæ for the LU-decomposition for the matrix $\mathcal{T}_n$. We also derive explicit formula for its inverse. Similarly to the results of [1, 2, 4, 6], the size of the matrix does not really matter, and one can think about an infinite matrix $\mathcal{T}$ and restrict it whenever necessary to the first n rows resp. columns and write $\mathcal{T}_n$. The entries of the inverse matrix $\mathcal{T}_n^{-1}$ are not closed form expressions, as in our previous paper [1, 2], but can only be given as a (simple) sum. We also provide the Cholesky decomposition. All the identities we will obtain hold for general q, and results about Lucas and Fibonacci numbers as well as Pell numbers etc., come out as corollaries for a special choice of q.

Firstly, we mention our general results depending on λ and then give their specializaitons for $\lambda = 1$. After that, we give examples of these results for the Lucas and Pell-Lucas numbers by taking special cases of q.

We will obtain the LU-decomposition $\mathcal{T} = L \cdot U$:

Theorem 1. *For $1 \le d \le n$ we have*

$$L_{n,d} = \mathbf{i}^{\lambda k(d-n)} q^{\frac{\lambda k(n-d)}{2}} \begin{bmatrix} 2d+k-1 \\ d \end{bmatrix}_{(-q^{\lambda+r};q^{\lambda})} \begin{bmatrix} n+d+k-1 \\ n \end{bmatrix}_{(-q^{\lambda+r};q^{\lambda})}^{-1} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_{(q^{\lambda};q^{\lambda})}.$$

Its generalized Fibonacci-Lucas corollary:

Corollary 1. *For $1 \le d \le n$,*

$$L_{n,d} = \left\langle \begin{matrix} 2d+k-1 \\ d \end{matrix} \right\rangle_{V(\lambda+r,\lambda)} \left\langle \begin{matrix} n+d+k-1 \\ n \end{matrix} \right\rangle_{V(\lambda+r,\lambda)}^{-1} \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\}_{U(\lambda)}.$$

As a consequence of Theorem 1 for $\lambda = 1$, we have

Corollary 2. *For $1 \le d \le n$,*

$$L_{n,d} = \mathbf{i}^{k(d-n)} q^{\frac{k(n-d)}{2}} \begin{bmatrix} 2d+r+k-1 \\ d+r \end{bmatrix}_{(-q;q)} \begin{bmatrix} n+d+r+k-1 \\ n+r \end{bmatrix}_{(-q;q)}^{-1} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_{(q;q)}.$$

In the $\lambda = 1$ case, its generalized Fibonacci-Lucas corollary:

Corollary 3. *For $1 \le d \le n$,*

$$L_{n,d} = \left\langle \begin{matrix} 2d+r+k-1 \\ d+r \end{matrix} \right\rangle_{V} \left\langle \begin{matrix} n+d+r+k-1 \\ n+r \end{matrix} \right\rangle_{V}^{-1} \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\}_{U}$$

From the corollaries above, we have the following examples: For $r = 1$ and $q = \left(1-\sqrt{5}\right) / \left(1+\sqrt{5}\right)$, we obtain a Fibonacci and Lucas consequence of Corollary 3:

$$L_{n,d} = \left\langle \begin{matrix} 2d+k \\ d+1 \end{matrix} \right\rangle_{L} \left\langle \begin{matrix} n+d+k \\ n+1 \end{matrix} \right\rangle_{L}^{-1} \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\}_{F}.$$

For $r = 0$ and $q = \left(1-\sqrt{2}\right) / \left(1+\sqrt{2}\right)$, we obtain a Pell and Pell-Lucas consequence of Corollary 3:

$$L_{n,d} = \left\langle \begin{matrix} 2d+k-1 \\ d \end{matrix} \right\rangle_{Q} \left\langle \begin{matrix} n+d+k-1 \\ n \end{matrix} \right\rangle_{Q}^{-1} \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\}_{P}.$$

Theorem 2. *For $1 \le d \le n$ we have*

$$U_{d,n} = (-1)^{d-1} \mathbf{i}^{-\lambda k(d+n) - \frac{\lambda k^2}{2} + \frac{\lambda k}{2} - kr} q^{\frac{\lambda k(d+n)}{2} + \frac{\lambda k(k-1)}{4} - \lambda d + \lambda d^2 + r(d-1) + \frac{rk}{2}}$$

$$\times \begin{bmatrix} d+n+k-1 \\ n \end{bmatrix}^{-1}_{(-q^{\lambda+r};q^{\lambda})} \begin{bmatrix} d+k-2 \\ k-1 \end{bmatrix}_{(q^{\lambda};q^{\lambda})} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_{(q^{\lambda};q^{\lambda})} \frac{(q^{\lambda};q^{\lambda})^2_{d-1}}{(-q^{\lambda+r};q^{\lambda})_{2d+k-2}}.$$

As a generalized Fibonacci-Lucas corollary of Theorem 2, we have

Corollary 4. *For* $1 \le d \le n$

$$U_{d,n} = (-1)^{(1+d)(1-\lambda+r)} (\alpha-\beta)^{2(d-1)} \left\langle \begin{matrix} d+n+k-1 \\ n \end{matrix} \right\rangle^{-1}_{V(\lambda+r,\lambda)}$$
$$\times \begin{Bmatrix} d+k-2 \\ k-1 \end{Bmatrix}_{U(\lambda)} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_{U(\lambda)} \left(\prod_{t=1}^{2d+k-2} V_{t\lambda+r} \right)^{-1} \left(\prod_{t=1}^{d-1} U_{t\lambda} \right)^2.$$

As a consequence of Theorem 2 for $\lambda = 1$, we have

Corollary 5. *For* $1 \le d \le n$

$$U_{d,n} = (-1)^{i-1} \mathbf{i}^{-k(d+n)-\frac{k^2}{2}+\frac{k}{2}-kr} q^{\frac{k(d+n)}{2}+\frac{k(k-1)}{4}-d+d^2+r(d-1)+\frac{rk}{2}}$$
$$\times \begin{bmatrix} 2d+r+k-2 \\ d-1 \end{bmatrix}^{-1}_{(-q;q)} \begin{bmatrix} d+n+r+k-1 \\ n+r \end{bmatrix}^{-1}_{(-q;q)}$$
$$\times \begin{bmatrix} 2d+k-2 \\ d-1 \end{bmatrix}_{(-q;q)} \begin{bmatrix} d+k-2 \\ k-1 \end{bmatrix}_{(q;q)} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_{(q;q)} \frac{(q;q)^2_{d-1}}{(-q;q)_{2d+k-2}}.$$

And its generalized Fibonacci-Lucas corollary:

Corollary 6. *For* $1 \le d \le n$

$$U_{d,n} = (-1)^{(d-1)r} (\alpha-\beta)^{2(d-1)}$$
$$\times \left\langle \begin{matrix} 2d+r+k-2 \\ d-1 \end{matrix} \right\rangle^{-1}_{V} \left\langle \begin{matrix} d+n+r+k-1 \\ n+r \end{matrix} \right\rangle^{-1}_{V} \left\langle \begin{matrix} 2d+k-2 \\ d-1 \end{matrix} \right\rangle_{V}$$
$$\times \begin{Bmatrix} d+k-2 \\ k-1 \end{Bmatrix}_{U} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_{U} \left(\prod_{t=1}^{d-1} U_t \right)^2 \left(\prod_{t=1}^{2d+k-2} V_t \right)^{-1}.$$

From the Corollaries above, we give the following examples:
For $r = 1$ and $q = \left(1-\sqrt{5}\right) / \left(1+\sqrt{5}\right)$, we obtain a Fibonacci and Lucas consequence of Corollary 6:

$$U_{d,n} = (-1)^{d-1} 5^{d-1} \left\langle \begin{matrix} 2d+k-1 \\ d-1 \end{matrix} \right\rangle^{-1}_{L} \left\langle \begin{matrix} d+n+k \\ n+1 \end{matrix} \right\rangle^{-1}_{L} \left\langle \begin{matrix} 2d+k-2 \\ d-1 \end{matrix} \right\rangle_{L}$$
$$\times \begin{Bmatrix} d+k-2 \\ k-1 \end{Bmatrix}_{F} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_{F} \left(\prod_{t=1}^{d-1} F_t \right)^2 \left(\prod_{t=1}^{2d+k-2} L_t \right)^{-1}.$$

For $r = 0$ and $q = \left(1-\sqrt{2}\right)/\left(1+\sqrt{2}\right)$, we obtain a Pell and Pell-Lucas consequence of Corollary 6:

$$U_{d,n} = 2^{d-1}\left\langle {2d+k-2 \atop d-1} \right\rangle_Q^{-1} \left\langle {d+n+k-1 \atop n} \right\rangle_Q^{-1} \left\langle {2d+k-2 \atop d-1} \right\rangle_Q \times \left\{ {d+k-2 \atop k-1} \right\}_P \left\{ {n-1 \atop d-1} \right\}_P \left(\prod_{t=1}^{d-1} P_t \right)^2 \left(\prod_{t=1}^{2d+k-2} Q_t \right)^{-1}.$$

We could also determine the inverses of the matrices L and U:

Theorem 3. *For $1 \le d \le n$ we have*

$$L^{-1}_{n,d} = \mathbf{i}^{-\lambda k(n-d)}(-1)^{n-d} q^{\frac{\lambda(n-d)(n-d+k-1)}{2}} \times \begin{bmatrix} 2n+k-2 \\ n \end{bmatrix}^{-1}_{\left(-q^{\lambda+r};q^{\lambda}\right)} \begin{bmatrix} n+d+k-2 \\ d \end{bmatrix}_{\left(-q^{\lambda+r};q^{\lambda}\right)} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_{\left(q^{\lambda};q^{\lambda}\right)}.$$

Its generalized Fibonacci-Lucas corollary:

Corollary 7. *For $1 \le d \le n$*

$$L^{-1}_{n,d} = \mathbf{i}^{\lambda\left(d^2+d-1-n\right)}(-1)^{n-d-\lambda nd} \times \left\langle {2n+k-2 \atop n} \right\rangle^{-1}_{V(\lambda+r,\lambda)} \left\langle {n+d+k-2 \atop d} \right\rangle_{V(\lambda+r,\lambda)} \left\{ {n-1 \atop d-1} \right\}_{U(\lambda,\lambda)}.$$

As a consequence of Theorem 3 for $\lambda = 1$, we have

Corollary 8. *For $1 \le d \le n$*

$$L^{-1}_{n,d} = \mathbf{i}^{-(k+2)(n-d)} q^{\frac{(n-d)(n-d+k-1)}{2}} \times \begin{bmatrix} n+d+r+k-2 \\ d+r \end{bmatrix}_{(-q;q)} \begin{bmatrix} 2n+r+k-2 \\ n+r \end{bmatrix}^{-1}_{(-q;q)} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_{(q;q)}.$$

Its generalized Fibonacci-Lucas corollary:

Corollary 9. *For $1 \le d \le n$*

$$L^{-1}_{n,d} = \mathbf{i}^{d(d+1)-n-1}(-1)^{d(n+1)-n} \times \left\langle {n+d+r+k-2 \atop d+r} \right\rangle_V \left\langle {2n+r+k-2 \atop n+r} \right\rangle^{-1}_V \left\{ {n-1 \atop d-1} \right\}_U.$$

Thus we have the following example: for $\lambda = 1$, $r = -2$ and $q = \left(1-\sqrt{5}\right)/\left(1+\sqrt{5}\right)$,

$$L^{-1}_{i,j} = \mathbf{i}^{j(j+1)-i-1}(-1)^{ij+j-i} \left\langle {i+j+k \atop j+2} \right\rangle_L \left\langle {2i+k \atop i+2} \right\rangle^{-1}_L \left\{ {i-1 \atop j-1} \right\}_F.$$

Theorem 4. *For $1 \le d \le n$ we have*

$$U_{d,n}^{-1} = \mathbf{i}^{\lambda k(d+n+r)+\lambda\binom{k}{2}}(-1)^{d-1}q^{\frac{-\lambda(n-d+k-1)(n+d)}{2}-\lambda dn-rn-\frac{\lambda k(k-1)}{4}-\frac{rk}{2}+r} \times \begin{bmatrix} n+d+k-2 \\ d-1 \end{bmatrix}_{(-q^{\lambda+r};q^{\lambda})} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_{(q^{\lambda};q^{\lambda})} \begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}_{(q^{\lambda};q^{\lambda})}^{-1} \times \frac{(-q^{\lambda+r};q^{\lambda})_{2n+k-1}}{(1+q^{\lambda d+r})} \frac{1}{(q^{\lambda};q^{\lambda})_{n-1}^2}.$$

And its generalized Fibonacci-Lucas corollary:

Corollary 10. *For $1 \le d \le n$*

$$U_{d,n}^{-1} = (-1)^{r(1-n)+(1+d)-dn\lambda}\,\mathbf{i}^{(d-1+(1-n)n+kr)\lambda-kr}\,(\alpha-\beta)^{-2(n-1)} \times \left\langle \begin{matrix} n+d+k-2 \\ d-1 \end{matrix} \right\rangle_{V(\lambda+r,\lambda)} \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\}_{U(\lambda)} \left\{ \begin{matrix} n+k-2 \\ k-1 \end{matrix} \right\}_{U(\lambda)}^{-1} \times \left(\prod_{t=1}^{2n+k-1} V_{t\lambda+r}\right)\left(\prod_{t=1}^{n-1} U_{t\lambda}\right)^{-2} \frac{1}{V_{\lambda d+r}}.$$

For $\lambda = 1$, as a consequence of Theorem 4, we have

Corollary 11. *For $1 \le d \le n$*

$$U_{d,n}^{-1} = \mathbf{i}^{k(d+n+r)+\binom{k}{2}}(-1)^{d-1}q^{\frac{-(n-d+k-1)(n+d)}{2}-dn-rn-\frac{k(k-1)}{4}-\frac{rk}{2}+r} \times \begin{bmatrix} 2n+r+k-1 \\ n \end{bmatrix}_{(-q;q)} \begin{bmatrix} 2n+k-2 \\ n \end{bmatrix}_{(-q;q)}^{-1} \begin{bmatrix} d+n+r+k-2 \\ d+r \end{bmatrix}_{(-q;q)} \times \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_{(q;q)} \begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}_{(q;q)}^{-1} \frac{(-q;q)_{2n+k-2}}{(q;q)_{n-1}^2}.$$

And its generalized Fibonacci-Lucas corollary:

Corollary 12. *For $1 \le d \le n$*

$$U_{d,n}^{-1} = (-1)^{d-1-dn+r-nr}\mathbf{i}^{d-n(n-1)-1}\,(\alpha-\beta)^{-2(n-1)} \times \left\langle \begin{matrix} 2n+r+k-1 \\ n \end{matrix} \right\rangle_{V} \left\langle \begin{matrix} 2n+k-2 \\ n \end{matrix} \right\rangle_{V}^{-1} \left\langle \begin{matrix} d+n+r+k-2 \\ d+r \end{matrix} \right\rangle_{V} \times \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\}_{U} \left\{ \begin{matrix} n+k-2 \\ k-1 \end{matrix} \right\}_{U}^{-1} \left(\prod_{t=1}^{n-1} U_t\right)^{-2} \left(\prod_{t=1}^{2n+k-2} V_t\right).$$

Especially for $\lambda = r = 1$ and $q = \left(1-\sqrt{5}\right)/\left(1+\sqrt{5}\right)$,

$$U_{d,n}^{-1} = (-1)^{d-dn-n}\mathbf{i}^{d-n(n-1)-1}5^{1-n}$$

$$\times \left\langle {2n+k \atop n} \right\rangle_L \left\langle {2n+k-2 \atop n} \right\rangle_L^{-1} \left\langle {d+n+k-1 \atop d+1} \right\rangle_L$$
$$\times \left\{ {n-1 \atop d-1} \right\}_F \left\{ {n+k-2 \atop k-1} \right\}_F^{-1} \left(\prod_{t=1}^{n-1} F_t \right)^{-2} \left(\prod_{t=1}^{2n+k-2} L_t \right).$$

As a consequence, we can compute the determinant of $\mathcal{T}_n$, since it is simply evaluated as $U_{1,1} \cdots U_{n,n}$:

Theorem 5.

$$\det \mathcal{T}_n = \mathbf{i}^{-\frac{\lambda k^2}{2}+\frac{\lambda k}{2}-kr+n(n+3)-k\lambda n(n+1)}$$
$$\times q^{\frac{\lambda n(n+1)(2n+1)}{6}+\frac{\lambda k(k-1)}{4}-r+\frac{rk}{2}+\frac{1}{2}n(n+1)(\lambda k-\lambda+r)} \frac{(q^\lambda;q^\lambda)_{d-1}^2}{(-q^{\lambda+r};q^\lambda)_{2d+k-2}}$$
$$\times \prod_{d=1}^{n} \begin{bmatrix} 2d+k-1 \\ d \end{bmatrix}_{(-q^{\lambda+r};q^\lambda)}^{-1} \begin{bmatrix} d+k-2 \\ k-1 \end{bmatrix}_{(q^\lambda;q^\lambda)}.$$

Its generalized Fibonacci and Lucas corollary

$$\det \mathcal{T}_n = \mathbf{i}^{(1-\lambda+r)n(n+3)} (\alpha-\beta)^{\frac{n(n-1)}{2}} \left(\prod_{t=1}^{2d+k-2} V_{t\lambda+r} \right)^{-1} \left(\prod_{t=1}^{d-1} U_{t\lambda} \right)^2$$
$$\times \prod_{d=1}^{n} \left\langle {2d+k-1 \atop d} \right\rangle_{V(\lambda+r,\lambda)}^{-1} \left\{ {d+k-2 \atop k-1} \right\}_{U(\lambda)}.$$

For $q = \left(1-\sqrt{5}\right)/\left(1+\sqrt{5}\right)$, $\lambda = 1$ and $r = 0$, we easily see that

$$\det \mathcal{T}_n = 5^{\frac{n(n-1)}{2}} \prod_{d=1}^{n} \left\langle {2d+k-2 \atop d-1} \right\rangle_L^{-1} \left\langle {2d+k-1 \atop d} \right\rangle_L^{-1}$$
$$\times \left\langle {2d+k-2 \atop d-1} \right\rangle_L \left\{ {d+k-2 \atop k-1} \right\}_F \left(\prod_{t=1}^{d-1} F_t \right)^2 \left(\prod_{t=1}^{2d+k-2} L_t \right)^{-1}.$$

Now we compute the inverse of the matrix $\mathcal{T}$. This time it depends on the dimension, so we compute $(\mathcal{T}_n)^{-1}$.

Theorem 6. *For* $1 \le i, j \le n$*:*

$$\left((\mathcal{T}_n)^{-1}\right)_{i,k} = \mathbf{i}^{\lambda\binom{k}{2}+\lambda k(i+r)+\lambda kj} (-1)^{i-1-j} q^{-\frac{\lambda k(k-1+2j)}{4}-\frac{rk}{2}+r-\frac{\lambda(i(k-1)+1)}{2}+\frac{\lambda j(1+j)}{2}}$$
$$\times \frac{1}{(-q^{\lambda+r};q^\lambda)_i (-q^{\lambda+r};q^\lambda)_{k-1} (-q^{\lambda+r};q^\lambda)_j (q^\lambda;q^\lambda)_{j-1} (q^\lambda;q^\lambda)_{i-1}}$$
$$\times \sum_{\max\{i,j\} \le h \le n} (-1)^h q^{-\frac{1}{2}h\lambda(j+2i)-rh} \frac{\left(-q^{\lambda+r};q^\lambda\right)_{h+i+k-2} \left(-q^{\lambda+r};q^\lambda\right)_{h+j+k-2}}{(q^\lambda;q^\lambda)_{h-i} (q^\lambda;q^\lambda)_{h-j}}$$

$$\times \frac{\left(1+q^{\lambda(2h+k-1)+r}\right)\left(1+q^{\lambda h+r}\right)}{\left(1+q^{\lambda(h+k-1)+r}\right)}.$$

Finally, we provide the Cholesky decomposition.

Theorem 7. *For $i, j \geq 1$:*

$$\begin{aligned}\mathcal{C}_{i,j} = \mathbf{i}^{-\frac{rk}{2}-\frac{\lambda k(k-1)}{4}+\lambda k i+j-1} q^{\frac{\lambda i k}{2}+\frac{\lambda j(j-1)}{2}+\frac{\lambda k(k-1)}{8}-\frac{r}{2}+\frac{rj}{2}+\frac{rk}{4}} \\ \times \begin{bmatrix} i+j+k-1 \\ i \end{bmatrix}^{-1}_{(-q^{\lambda+r};q^{\lambda})} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{(q^{\lambda};q^{\lambda})} (q^{\lambda};q^{\lambda})_{j-1} \\ \times \sqrt{\begin{bmatrix} 2j+k-1 \\ j \end{bmatrix}_{(-q^{\lambda+r};q^{\lambda})} \frac{1}{(-q^{\lambda+r};q^{\lambda})_{2j+k-2}} \begin{bmatrix} j+k-2 \\ k-1 \end{bmatrix}_{(q^{\lambda};q^{\lambda})}}.\end{aligned}$$

Its generalized Fibonacci-Lucas Corollary:

Corollary 13. *For $i, j \geq 1$:*

$$\begin{aligned}\mathcal{C}_{i,j} = \mathbf{i}^{(j-1)(r+j\lambda+1)} (-1)^{ik\lambda} (\alpha-\beta)^{j-1} \\ \times \left\langle \begin{matrix} i+j+k-1 \\ i \end{matrix} \right\rangle^{-1}_{V(\lambda+r,\lambda)} \left\{ \begin{matrix} i-1 \\ j-1 \end{matrix} \right\}_{U(\lambda)} \left(\prod_{t=1}^{j-1} U_{t\lambda} \right) \\ \times \sqrt{\left\langle \begin{matrix} 2j+k-1 \\ j \end{matrix} \right\rangle_{V(\lambda+r,\lambda)} \left(\prod_{t=1}^{2j+k-2} V_{t\lambda+r} \right)^{-1} \left\{ \begin{matrix} j+k-2 \\ k-1 \end{matrix} \right\}_{U(\lambda)}}.\end{aligned}$$

2. Proofs

We will get relavent quantites related with the LU-decomposition by our usual guessing strategy. As already mentioned, we will evaluate the relevant sums with the q-Zeilberger algorithm, in particular the version that was developed at the RISC in Linz [5].

First, we show that $\sum_j L_{m,j}U_{j,n}$ is indeed the matrix $\mathcal{T}$. We compute

$$\begin{aligned}\sum_j L_{m,j}U_{j,n} = \sum_j \mathbf{i}^{\lambda k(j-m)} q^{\frac{\lambda k(m-j)}{2}} \frac{(-q^{\lambda+r};q^{\lambda})_{2j+k-1}(-q^{\lambda+r};q^{\lambda})_m}{(-q^{\lambda+r};q)_j(-q^{\lambda+r};q^{\lambda})_{m+j+k-1}} \frac{(q^{\lambda};q^{\lambda})_{m-1}}{(q^{\lambda};q^{\lambda})_{j-1}(q^{\lambda};q^{\lambda})_{m-j}} \\ \times (-1)^{j-1}\mathbf{i}^{-\lambda k(j+n)-\frac{\lambda k^2}{2}+\frac{\lambda k}{2}-kr} q^{\frac{\lambda k(j+n)}{2}+\frac{\lambda k(k-1)}{4}-\lambda j+\lambda j^2+r(j-1)+\frac{rk}{2}} \\ \times \frac{(-q^{\lambda+r};q^{\lambda})_n(-q^{\lambda+r};q^{\lambda})_{j+k-1}}{(-q^{\lambda+r};q^{\lambda})_{2j+k-2}(-q^{\lambda+r};q^{\lambda})_{j+n+k-1}} \frac{(q^{\lambda};q^{\lambda})_{j+k-2}(q^{\lambda};q^{\lambda})_{n-1}}{(q^{\lambda};q^{\lambda})_{n-j}(q^{\lambda};q^{\lambda})_{k-1}}.\end{aligned}$$

We only keep terms that do contain the summation index j:

$$\begin{aligned}\sum_j (-1)^j q^{-\lambda j+\lambda j^2+rj} \frac{(-q^{\lambda+r};q^{\lambda})_{2j+k-1}}{(-q^{\lambda+r};q^{\lambda})_j(-q^{\lambda+r};q^{\lambda})_{m+j+k-1}(q^{\lambda};q^{\lambda})_{j-1}(q^{\lambda};q^{\lambda})_{m-j}} \\ \times \frac{(-q^{\lambda+r};q^{\lambda})_{j+k-1}}{(-q^{\lambda+r};q^{\lambda})_{2j+k-2}(-q^{\lambda+r};q^{\lambda})_{j+n+k-1}} \frac{(q^{\lambda};q^{\lambda})_{j+k-2}}{(q^{\lambda};q^{\lambda})_{n-j}}.\end{aligned}$$

We set $q^\lambda = Q$ and $r = s\lambda$ and pull out an irrelevant factor:

$$\sum_j (-1)^j Q^{-j+j^2+sj} \frac{(-Q^s;Q)_{2j+k}}{(-Q^s;Q)_{j+1}(-Q^s;Q)_{m+j+k}} \times \frac{(-Q^s;Q)_{j+k}}{(-Q^s;Q)_{2j+k-1}(-Q^s;Q)_{j+n+k}} \frac{(Q;Q)_{j+k-2}}{(Q;Q)_{n-j}(Q;Q)_{j-1}(Q;Q)_{m-j}}.$$

If we consider the sum as a function of m, computer algebra produces for $m \ge 2$ the recursion

$$\mathsf{SUM}_m = \frac{1+Q^{m+n+s-1}}{(1-Q^{m-1})(1+Q^{m+s})(1+Q^{k+m+n+s-1})}\mathsf{SUM}_{m-1}.$$

Since

$$\mathsf{SUM}_1 = -Q^s \frac{(Q;Q)_{k-1}}{(-Q^s;Q)_2(-Q^s;Q)_{1+n+k}(Q;Q)_{n-1}},$$

we get a product representation for SUM_m, and together with the irrelevant factors that we dropped on the way, the terms from the matrix $\mathcal{T}$.

Now we look at the inverse matrices:

$$\sum_{n\le j\le m} L_{m,j}L^{-1}_{j,n}$$
$$= \sum_{n\le j\le m} \mathbf{i}^{\lambda k(j-m)} q^{\frac{\lambda k(m-j)}{2}} \frac{(-q^{\lambda+r};q^\lambda)_{2j+k-1}(-q^{\lambda+r};q^\lambda)_m}{(-q^{\lambda+r};q)_j(-q^{\lambda+r};q^\lambda)_{m+j+k-1}} \frac{(q^\lambda;q^\lambda)_{m-1}}{(q^\lambda;q^\lambda)_{j-1}(q^\lambda;q^\lambda)_{m-j}}$$
$$\times\, \mathbf{i}^{-\lambda k(j-n)}(-1)^{j-n} q^{\frac{\lambda(j-n)(j-n+k-1)}{2}} \frac{(-q^{\lambda+r};q^\lambda)_{j+n-2+k}(-q^{\lambda+r};q^\lambda)_j}{(-q^{\lambda+r};q^\lambda)_{2j+k-2}(-q^{\lambda+r};q^r)_n} \frac{(q^\lambda;q^\lambda)_{j-1}}{(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{j-n}}.$$

Again, we drop all the terms that do not depend on j:

$$\sum_{n\le j\le m} (-1)^j q^{\lambda\binom{j}{2}-\lambda jn} \frac{(-q^{\lambda+r};q^\lambda)_{2j+k-1}(-q^{\lambda+r};q^\lambda)_{j+n+k-2}}{(-q^{\lambda+r};q^\lambda)_{m+j+k-1}(-q^{\lambda+r};q^\lambda)_{2j+k-2}} \frac{1}{(q^\lambda;q^\lambda)_{m-j}(q^\lambda;q^\lambda)_{j-n}}.$$

After the substitutions,

$$\sum_{n\le j\le m} (-1)^j Q^{\binom{j}{2}-jn} \frac{(-Q;Q)_{2j+k+s-1}(-Q;Q)_{j+n+k+s-2}}{(-Q;Q)_{m+j+k+s-1}(-Q;Q)_{2j+k+s-2}} \frac{1}{(Q;Q)_{m-j}(Q;Q)_{j-n}}.$$

Computer algebra tells us that this is 0, for $m \ne n$, as required. The value 1 for $m = n$ can be computed by hand.

Now we consider the other inverse matrix:

$$\sum_{m\le j\le n} U_{m,j}U^{-1}_{j,n}$$
$$= \sum_{m\le j\le n} (-1)^{m-1} \mathbf{i}^{-\lambda k(m+j)-\frac{\lambda k^2}{2}+\frac{\lambda k}{2}-kr} q^{\frac{\lambda k(m+j)}{2}+\frac{\lambda k(k-1)}{4}-\lambda m+\lambda m^2+r(m-1)+\frac{rk}{2}}$$
$$\times \frac{(-q^{\lambda+r};q^\lambda)_j(-q^{\lambda+r};q^\lambda)_{m+k-1}}{(-q^{\lambda+r};q^\lambda)_{2m+k-2}(-q^{\lambda+r};q^\lambda)_{m+j+k-1}} \frac{(q^\lambda;q^\lambda)_{m+k-2}(q^\lambda;q^\lambda)_{j-1}}{(q^\lambda;q^\lambda)_{j-m}(q^\lambda;q^\lambda)_{k-1}}$$

$$\times\, \mathbf{i}^{\lambda k(j+n+r)+\lambda\binom{k}{2}}(-1)^{j-1}q^{\frac{-\lambda(n-j+k-1)(n+j)}{2}-\lambda jn-rn-\frac{\lambda k(k-1)}{4}-\frac{rk}{2}+r}$$
$$\times\, \frac{(-q^{\lambda+r};q^\lambda)_{2n+k-1}(-q^{\lambda+r};q^\lambda)_{n+j+k-2}}{(-q^{\lambda+r};q^\lambda)_j(-q^{\lambda+r};q^\lambda)_{n+k-1}}\frac{(q^\lambda;q^\lambda)_{k-1}}{(q^\lambda;q^\lambda)_{j-1}(q^\lambda;q^\lambda)_{n-j}(q^\lambda;q^\lambda)_{n+k-2}}.$$

Again, we only keep factors that depend on j:

$$\sum_{m\le j\le n}(-1)^j q^{\lambda\binom{j+1}{2}-\lambda jn}\frac{(-q^{\lambda+r};q^\lambda)_{n+j+k-2}}{(-q^{\lambda+r};q^\lambda)_{m+j+k-1}}\frac{1}{(q^\lambda;q^\lambda)_{n-j}(q^\lambda;q^\lambda)_{j-m}}.$$

After substitutions,

$$\sum_{m\le j\le n}(-1)^j Q^{\binom{j+1}{2}-jn}\frac{(-Q;Q)_{n+j+k+s-2}}{(-Q;Q)_{m+j+k+s-1}}\frac{1}{(Q;Q)_{n-j}(Q;Q)_{j-m}},$$

and computer algebra evaluates this again to 0 for $m\neq 0$.

Finally, for the Cholesky decomposition, we need to consider

$$\sum_{1\le j\le\min\{i,l\}}\mathcal{C}_{i,j}\mathcal{C}_{l,j}=\sum_{1\le j\le\min\{i,l\}}q^{\frac{\lambda ik}{2}+\frac{\lambda j(j-1)}{2}+\frac{\lambda k(k-1)}{8}-\frac{r}{2}+\frac{rj}{2}+\frac{rk}{4}}\mathbf{i}^{-\frac{rk}{2}-\frac{\lambda k(k-1)}{4}+\lambda ki+j-1}$$
$$\times\,\frac{(-q^{\lambda+r};q^\lambda)_i}{(-q^{\lambda+r};q^\lambda)_{i+j+k-1}}\frac{(q^\lambda;q^\lambda)_{i-1}}{(q^\lambda;q^\lambda)_{i-j}}$$
$$\times\,\frac{(1+q^{r+\lambda(2j+k-1)})(-q^{\lambda+r};q^\lambda)_{j+k-1}}{(-q^{\lambda+r};q^\lambda)_j}\frac{(q^\lambda;q^\lambda)_{j+k-2}}{(q^\lambda;q^\lambda)_{k-1}(q^\lambda;q^\lambda)_{j-1}}$$
$$\times\, q^{\frac{\lambda lk}{2}+\frac{\lambda j(j-1)}{2}+\frac{\lambda k(k-1)}{8}-\frac{r}{2}+\frac{rj}{2}+\frac{rk}{4}}\mathbf{i}^{-\frac{rk}{2}-\frac{\lambda k(k-1)}{4}+\lambda kl+j-1}$$
$$\times\,\frac{(-q^{\lambda+r};q^\lambda)_l}{(-q^{\lambda+r};q^\lambda)_{l+j+k-1}}\frac{(q^\lambda;q^\lambda)_{l-1}}{(q^\lambda;q^\lambda)_{l-j}}.$$

The terms that depend on j:

$$\sum_{1\le j\le\min\{i,l\}}(-1)^j q^{\lambda j(j-1)+rj}$$
$$\times\,\frac{(1+q^{r+\lambda(2j+k-1)})(-q^{\lambda+r};q^\lambda)_{j+k-1}}{(-q^{\lambda+r};q^\lambda)_j(-q^{\lambda+r};q^\lambda)_{i+j+k-1}(-q^{\lambda+r};q^\lambda)_{l+j+k-1}}\frac{(q^\lambda;q^\lambda)_{j+k-2}}{(q^\lambda;q^\lambda)_{i-j}(q^\lambda;q^\lambda)_{j-1}(q^\lambda;q^\lambda)_{l-j}}.$$

After the substitutions,

$$\sum_{1\le j\le\min\{i,l\}}(-1)^j Q^{j(j-1)+sj}$$
$$\times\,\frac{(1+Q^{s+2j+k-1})(-Q;Q)_{j+k+s-1}}{(-Q;Q)_{j+s}(-Q;Q)_{i+j+k+s-1}(-Q;Q)_{l+j+k+s-1}}\frac{(Q;Q)_{j+k-2}}{(Q;Q)_{i-j}(Q;Q)_{j-1}(Q;Q)_{l-j}}.$$

Computer algebra produces the recursion (for $i\ge 2$)

$$\mathsf{SUM}_i=\frac{1+Q^{i+l+s-1}}{(1-Q^{i-1})(1+Q^{i+s})(1+Q^{i+k+l+s-1})}\mathsf{SUM}_{i-1}.$$

The initial value is easily found:

$$\mathsf{SUM}_1 = -\frac{Q^s}{(-Q;Q)_{1+s}(-Q;Q)_{l+k+s}}\frac{(Q;Q)_{k-1}}{(Q;Q)_{l-1}}.$$

Iteration gives the product form for SUM_i, and together with the dropped factors we get the correct terms $t_{i,l}$ of the matrix $\mathcal{T}$.

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