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# Domination type parameters of Pell graphs*

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**Abstract**

Pell graphs are defined on certain ternary strings as special subgraphs of Fibonacci cubes of odd index. In this work the domination number, total domination number, 2-packing number, connected domination number, paired domination number, and signed domination number of Pell graphs are studied. Using integer linear programming, exact values and some estimates for these numbers of small Pell graphs are obtained. Furthermore, some theoretical bounds are obtained for the domination numbers and total domination numbers of Pell graphs.

*Keywords: Pell graphs, Fibonacci cube, domination number, integer linear programming.*

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## 1 Introduction

One of the basic models for interconnection networks is the $n$-dimensional hypercube graph $Q_n$. It has $2^n$ vertices, represented by all binary strings of length $n$, and two vertices in $Q_n$ are adjacent if they differ in exactly one coordinate. For convenience, we set

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$Q_0 = K_1$. The $n$ dimensional *Fibonacci cube* $\Gamma_n$ is defined as the subgraph of $Q_n$ induced by the vertices whose string representations are Fibonacci strings. They were introduced by Hsu [10] as an alternative model for interconnection networks and extensively studied in the literature [13]. There are numerous subgraphs and variants of Fibonacci cubes in the literature, such as Lucas cubes [15], generalized Fibonacci cubes [11], $k$-Fibonacci cubes [5] and Pell graphs [14].

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A set $D \subseteq V$ is called a *dominating set* of $G$ if every vertex in $V \setminus D$ is adjacent to some vertex in $D$. Then the *domination number* $\gamma(G)$ of $G$ is defined as the minimum cardinality of a dominating set of $G$. Similarly, a set $D \subseteq V$ is called a *total dominating set* of a graph $G$ without isolated vertices, if every vertex in $V$ is adjacent to some vertex in $D$ and the *total domination number* $\gamma_t(G)$ of $G$ is defined as the minimum cardinality of a total dominating set of $G$.

The domination type parameters of Fibonacci and Lucas cubes are first considered in [3, 17]. Using integer linear programming, domination and total domination numbers of these cubes and some additional domination type parameters of these cubes [2, 12] and hypercubes [2] are considered in the literature. Furthermore, upper bounds and lower bounds on domination and total domination numbers of Fibonacci and Lucas cubes are obtained in [2, 3, 17, 18, 19, 20]. The domination and total domination number of $k$-Fibonacci cubes are considered in [6]. In this work, we studied some domination type parameters of Pell graphs.

## 2 Preliminaries

Let $f_n$ denote the Fibonacci numbers defined as $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Similarly, let $p_n$ denote the Pell numbers defined as $p_0 = 1$, $p_1 = 2$ and $p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$. Here we remark that the generating function of $p_n$ (see, for example [9]) is

$$\sum_{n \geq 0} p_n x^n = \frac{1}{1 - 2x - x^2}. \tag{2.1}$$

Binary strings of length $n$ not containing two consecutive 1s constitute the set of *Fibonacci strings* $\mathcal{F}_n$ of length $n$, that is, the binary strings $b_1 b_2 \dots b_n$ such that $b_i \cdot b_{i+1} = 0$ for all $i = 1, 2, \dots, n-1$.

Ternary strings over the alphabet $\{0, 1, 2\}$ where there are no maximal blocks of 2s of odd length constitute the set of *Pell strings*, $\mathcal{P}_n$. Then the $n$ dimensional Pell graph, $\Pi_n$, is defined as the simple graph where the vertices are represented by the Pell strings of length $n$, and two vertices are adjacent whenever one of them can be obtained from the other by replacing a 0 with a 1 (or vice versa), or by replacing a factor 11 with 22 (or vice versa) [14]. The vertices of $\Pi_n$ can be partitioned into vertices that start with 0, vertices that start with 1 and vertices that start with 22. The subgraphs induced by these vertices are isomorphic to $\Pi_{n-1}$, $\Pi_{n-1}$, and $\Pi_{n-2}$, respectively. This gives the following canonical decomposition of Pell graphs for $n \geq 2$

$$\Pi_n = 0\Pi_{n-1} + 1\Pi_{n-1} + 22\Pi_{n-2}, \tag{2.2}$$

where $\Pi_0 = K_1$ and $\Pi_1 = K_2$. Here remark that we have also to add the edges of perfect matchings between $0\Pi_{n-1}$ and $1\Pi_{n-1}$; and also between $22\Pi_{n-2}$ and $11\Pi_{n-1}$ (an induced subgraph of $1\Pi_{n-1}$).

Every Pell string decomposes uniquely into the product of the factors $0, 1$ and $22$. Let $\psi\colon\ \mathcal{P}_n \to \mathcal{F}_{2n}$ where $\psi(0) = 10$, $\psi(1) = 00$ and $\psi(22) = 0100$. Hence, we know that $\psi$ maps any Pell string of length $n$ to a unique Fibonacci string of length $2n$ with no $0101$ factors and without a final $1$, which are called *Pell binary strings*. For a graph $G$, we denote a subgraph $H$ of $G$ by $H \subseteq G$. Then using this notation and the $\psi$ mapping it is shown that

**Theorem 2.1** ([14, Theorem 7])**.** *For $n \geq 1$, we have the inclusion $\Pi_n \subseteq \Gamma_{2n-1}$.*

Let $\Gamma^*_{2n}$ be the Hamming graph generated by the set of all Pell binary strings of length $2n$ then we have the following result showing that $\Pi_n$ is isomorphic to an induced subgraph of $\Gamma_{2n-1}0$.

**Theorem 2.2** ([14, Theorem 8])**.** *The graphs $\Pi_n$ and $\Gamma^*_{2n}$ are isomorphic.*

Let $N(v)$ denote the open neighborhood of $v \in V$, that is, the set of vertices adjacent to $v$, and $N[v] = N(v) \cup \{v\}$. Using Theorem 2.1, we have the following Lemma.

**Lemma 2.3.** *Let $v \in \Pi_n \subseteq \Gamma_{2n-1}0$. For any $u \in N(v) \subseteq \Gamma_{2n-1}0$, the binary string representation of $u$ can not have two non-overlapping $0101$ factors as a substring.*

*Proof.* Assume that there is a vertex $u \in N(v)$ of the form $\alpha_1 0101 \alpha_2 0101 \alpha_3 0 \in \mathcal{F}_{2n-1}0$. Then we know that the distance between $u$ and $v$ in $\Gamma_{2n-1}$ is $1$. Hence, $v$ should have a $0101$ factor, which is a contradiction. □

Let $\alpha 0(0101)0\beta \in \mathcal{F}_{2n}$ for some Fibonacci strings $\alpha$ and $\beta$ which do not have a $0101$ factor. Let us define the maps $\phi_1$, $\phi_2$ and $\phi$ from $\mathcal{F}_{2n}$ into $\mathcal{F}_{2n}$ by setting

$$
\begin{aligned}
\phi_1(\alpha 0(0101)0\beta) &= \alpha 0(0001)0\beta, \\
\phi_2(\alpha 0(0101)0\beta) &= \alpha 0(0100)0\beta, \\
\phi(\alpha 0(0101)0\beta) &= \alpha 0(0000)0\beta.
\end{aligned}
$$

## 3 Main results

We first interrelate the domination and total domination numbers of Fibonacci cubes and Pell graphs using Theorem 2.1 and Lemma 2.3.

**Proposition 3.1.** *For any positive integer $n$, we have*

(i) $\gamma(\Pi_n) \leq \gamma(\Gamma_{2n-1})$

(ii) $\gamma_t(\Pi_n) \leq \gamma_t(\Gamma_{2n-1})$

*Proof.* (i) Let $D$ be a minimal dominating set of $\Gamma_{2n-1}$ and set

$$
\begin{aligned}
D' = \{\alpha \mid \alpha \text{ is a Pell binary string from } D0\} \cup \\
\cup \{\phi(\beta 0) \mid \beta 0 \in D0 \text{ has one } 0101 \text{ factor}\}.
\end{aligned}
$$

Note that $|D'| \leq |D|$. Let $u$ be a vertex of $\Pi_n$. Then the vertex $\psi(u)$ is dominated in $\Gamma_{2n-1}0$ by some $d0 \in D0$. If $d0$ is a Pell binary string then $d0$ belongs to $D'$. If $d0$ is not a Pell binary string then we know that it has only one $0101$ factor and $\psi(u)$ must be of the form $\phi_1(d0)$ or $\phi_2(d0)$, which are also dominated by a Pell binary string $\phi(d0)$. Then we obverse that $D'$ is a dominating set of $\Pi_n$. Hence, we have $\gamma(\Pi_n) \leq \gamma(\Gamma_{2n-1})$.

(ii) Using the same argument in the previous part, assume that $D$ is a minimal total dominating set of $\Gamma_{2n-1}$. Then we merely need to show that $D'$ is a total dominating set. Since $D$ is a total dominating set in $\Gamma_{2n-1}$, we know that every vertex $v \in V(\Pi_n) \subseteq V(\Gamma_{2n-1}0)$ must be adjacent to some vertex $w \in D0$. If $w \in D'$, there is nothing to show. Otherwise, $w$ must have one 0101 factor. Since Pell binary string representations of the vertices in $\Pi_n$ do not have a 0101 factor, $v \in V(\Pi_n)$ must be of the form $\phi_1(w)$ or $\phi_2(w)$. Hence, $v$ is also adjacent to $\phi(w) \in D'$. □

Using the canonical decomposition (2.2) of $\Pi_n$, we obtain the following results.

**Proposition 3.2.** *For any integer $n \geq 3$, we have*

(i) $\gamma(\Pi_n) \leq 2\gamma(\Pi_{n-1}) + \gamma(\Pi_{n-2})$

(ii) $\gamma_t(\Pi_n) \leq 2\gamma(\Pi_{n-1}) + \gamma_t(\Pi_{n-2})$

(iii) $\gamma(\Pi_n) \leq \gamma_t(\Pi_n) \leq 5\gamma(\Pi_{n-2}) + 2\gamma(\Pi_{n-3})$

*Proof.* (i) This follows directly from the canonical decomposition (2.2) of Pell graphs.

(ii) Let $D_1$ be a dominating set for $\Pi_{n-1}$ and $D_2$ be a total dominating set for $\Pi_{n-2}$. From (2.2) we know that there is a perfect matching between $0\Pi_{n-1}$ and $1\Pi_{n-1}$. Using this perfect matching, we conclude that the set $0D_1 \cup 1D_1 \cup 22D_2$ is a total dominating set for $\Pi_n$, which gives the desired result.

(iii) This follows from using the canonical decomposition (2.2) of Pell graphs recursively and the perfect matchings between the induced subgraphs, namely 5 copies of $\Pi_{n-2}$ and 2 copies of $\Pi_{n-3}$. □

Considering the vertices of high degrees, lower bounds on $\gamma(\Gamma_n)$ and $\gamma(\Lambda_n)$ are obtained in [17, Theorem 3.2] and [3, Theorem 3.5.], respectively. Using the same argument, we obtain the lower bound for $\gamma(\Pi_n)$ in Proposition 3.4. Before we introduce this lower bound, we have the following remark on the degree distribution of the vertices of $\Pi_n$.

**Remark 3.3.** We know that $\Pi_n$ is an induced subgraph of $\Gamma_{2n-1}0$, which means that the degrees of the vertices of $\Pi_n$ is at most $2n - 1$. It is shown in [14, Proposition 27] that $1^n$ is the unique vertex having degree $2n - 1$ for $n \geq 2$. Using the recursive relation in [14, Theorem 29], which gives the number of all vertices of $\Pi_n$ having fixed degree, it is easy to show that for $n \geq 3$, there are only 2 vertices having degree $2n - 2$ (namely, $01^{n-1}$ and $1^{n-1}0$), and for $n \geq 4$ there are exactly $n + 1$ vertices having degree $2n - 3$. The rest of the vertices of $\Pi_n$ have degree at most $2n - 4$ for $n \geq 4$.

**Proposition 3.4.** *For any $n \geq 7$, we have $\gamma_t(\Pi_n) \geq \gamma(\Pi_n) \geq \left\lceil \dfrac{p_n - n - 8}{2n - 3} \right\rceil$.*

*Proof.* Let $D$ be a minimum dominating set of $\Pi_n$ and define the over domination of $\Pi_n$ with respect to $D$ as

$$OD(\Pi_n) = \left( \sum_{v \in D} \left( \deg(v) + 1 \right) \right) - |V(\Pi_n)| .$$

Let $S = \{v \in V(\Pi_n) \mid \deg(v) \geq 2n-3\}$. Using Remark 3.3, we have

$$\begin{aligned} 0 \leq OD(\Pi_n) &= 2n + 2(2n-1) + (n+1)(2n-2) - p_n + \sum_{v \in D \setminus S} \big(\deg(v)+1\big) \\ &\leq 2n^2 + 6n - 4 - p_n + (|D| - |S|)(2n-3) \\ &= n + 8 - p_n + |D|(2n-3) \end{aligned}$$

which gives the desired result. □

### 3.1 Integer linear programming for domination numbers

Suppose each vertex $v \in V(\Pi_n)$ is associated with a binary variable $x_v$. The problems of determining $\gamma(\Pi_n)$ and $\gamma_t(\Pi_n)$ can be expressed as problems of minimizing the objective function

$$\sum_{v \in V(\Pi_n)} x_v \tag{3.1}$$

subject to the following constraints for every $v \in V(\Lambda_n)$:

$$\sum_{a \in N[v]} x_a \geq 1 \text{ (for domination number)},$$
$$\sum_{a \in N(v)} x_a \geq 1 \text{ (for total domination number)}.$$

The value of the objective function (3.1) gives $\gamma(\Pi_n)$ and $\gamma_t(\Pi_n)$, respectively. Note that this problem has $p_n$ binary variables and $p_n$ constraints.

We implemented the integer linear programming problem (3.1) on Intel Core i7-10875H CPU @ 2.30GHz with 32GB RAM running the Ubuntu 20.04 LTS Linux operating system and using Gurobi Optimizer [8]. We obtain the exact values of $\gamma(\Pi_n)$ for $n \leq 6$ and $\gamma_t(\Pi_n)$ for $n \leq 7$. Furthermore, we obtain the estimates $60 \leq \gamma(\Pi_7) \leq 64$ (takes approximately 1 hour) and $137 \leq \gamma(\Pi_7) \leq 162$ (takes approximately 1 hour) . We collect the values of $\gamma(\Pi_n)$ and $\gamma_t(\Pi_n)$ that we obtained from (3.1) in Table 1. In Tables 2 and 3 we present examples of a minimal dominating and total dominating sets that were obtained during the computation of these values. We also present an example of a dominating set of $\Pi_7$ having cardinality 64 in Appendix (see, Table 11).

Table 1: Domination and total domination numbers for small Pell graphs.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\lvert V(\Pi_n)\rvert$ | 2 | 5 | 12 | 29 | 70 | 169 | 408 | 985 |
| $\gamma(\Pi_n)$ | 1 | 2 | 4 | 7 | 14 | 30 | 60−64 | |
| $\gamma_t(\Pi_n)$ | 2 | 2 | 4 | 9 | 16 | 34 | 72 | 137−162 |

Using the computation results presented in Table 1, Proposition 3.2 and a simple induction argument we obtain the following results.

**Theorem 3.5.** *For $n \geq 6$, we have $\gamma(\Pi_n) \leq 22p_{n-4} - 40p_{n-5}$ ; and for $n \geq 9$, we have $\gamma_t(\Pi_n) \leq 22p_{n-4} - 40p_{n-5}$ .*

*Proof.* From Proposition 3.2 and Table 1, we know that

$$\gamma(\Pi_n) \leq 2\gamma(\Pi_{n-1}) + \gamma(\Pi_{n-2}) \tag{3.2}$$

and $\gamma(\Pi_6) = 30$, $\gamma(\Pi_7) \leq 64$. We set $s_6 = 30$, $s_7 = 64$ and $s_n = 2s_{n-1} + s_{n-2}$ for $n \geq 8$. Using (3.2), one can easily see that $\gamma(\Pi_n) \leq s_n$ for $n \geq 6$. Let $S = \sum_{n\geq 0} s_{n+6}x^n$ be the generating function of the sequence $s_{n+6}$. Therefore, $S$ satisfies

$$S - 30 - 64x = 2x(S - 30) + x^2 S$$

which gives

$$S = \frac{30 + 4x}{1 - 2x - x^2}\,.$$

Then using (2.1), we obtain $s_{n+7} = 30p_{n+1} + 4p_n$ for $n \geq 0$ and $s_6 = 30p_0$. This is equivalent to $s_n = 22p_{n-4} - 40p_{n-5}$ for all $n \geq 6$. Using a similar argument, we obtain the desired result for the total domination number. □

**Remark 3.6.** For any graph $G$ of minimum degree $\delta$, a general upper bound due to Arnautov [1] and Payan [16] is

$$\gamma(G) \leq \frac{|V(G)|}{\delta + 1} \sum_{j=1}^{\delta+1} \frac{1}{j}\,. \tag{3.3}$$

We know that $\delta(\Pi_n) = \lceil \frac{n}{2} \rceil$ (cf. [14, Proposition 27]). Computing the upper bound in Theorem 3.5 and the right-hand side of the bound (3.3) for $\gamma(\Pi_n)$, we observe that our bound from Theorem 3.5 is better than the bound from (3.3) for $n \leq 44$.

Table 2: Example of a minimal dominating set for $\Pi_6$.

| |
|---|
| 000000, 000221, 001022, 001101, 001110, 001122, |
| 010011, 010110, 012200, 022000, 022111, 022220, |
| 100011, 100220, 101100, 102211, 110101, 110122, |
| 111001, 111010, 111221, 112211, 112222, 122022, |
| 122100, 220000, 220022, 220220, 221111, 222200. |

Table 3: Example of a minimal total dominating set for $\Pi_7$.

| |
|---|
| 0000000, 0001022, 0001122, 0001220, 0001221, 0002211, 0010000, 0010011, |
| 0010111, 0011100, 0012211, 0022011, 0022100, 0100101, 0100111, 0101010, |
| 0101101, 0110220, 0111220, 0122011, 0122111, 0122220, 0220111, 0220122, |
| 0221000, 0221001, 0221122, 0222200, 0222210, 1000110, 1000111, 1001001, |
| 1002200, 1010111, 1011001, 1011010, 1011110, 1022111, 1022122, 1022221, |
| 1100022, 1100220, 1101010, 1102200, 1102210, 1102222, 1110001, 1110010, |
| 1110022, 1110100, 1110220, 1111022, 1112201, 1112222, 1122000, 1122001, |
| 1220010, 1220100, 1221111, 1221221, 2200000, 2200111, 2201000, 2201111, |
| 2201221, 2210001, 2210111, 2211022, 2211110, 2212201, 2222110, 2222111. |

### 3.2 Additional domination type parameters of small Pell graphs

By using the integer linear programming approach several additional parameters of small Fibonacci cubes, Lucas cubes and $k$-Fibonacci cubes are obtained in [2, 6, 12, 20]. In this section we use a similar approach to obtain domination type parameters of small Pell graphs. For completeness of the paper, we first give the definition of these parameters and corresponding linear optimization problems similar to (3.1).

A set $X \subseteq V$ is a 2-packing if the distance $d(u,v) \geq 3$ for any $u, v \in X$, $u \neq v$. The maximum size of a 2-packing of $G$ is the 2-packing number of $G$ denoted $\rho(G)$. It can be determined using the following optimization problem:

$$\begin{aligned} \rho(G) = \max \sum_{v \in V} x_v & \\ \text{subject to} \quad \sum_{u \in N[v]} x_u \leq 1, & \text{ for all } v \in V. \end{aligned}$$

The independent domination number $i(G)$ is the minimum size of a dominating set that induces no edges (or, equivalently, the size of the smallest maximal independent set), which can be determined using the following optimization problem:

$$\begin{aligned} i(G) = \min \sum_{v \in V} x_v & \\ \text{subject to} \quad \sum_{u \in N[v]} x_u \geq 1, & \text{ for all } v \in V \\ (|V|-1)x_v + \sum_{u \in N(v)} x_u \leq |V|-1, & \text{ for all } v \in V. \end{aligned}$$

A set $X \subseteq V$ is a $k$-tuple dominating set of $G$ if for every vertex $v \in V$ we have $|N[v] \cap X| \geq k$, that is, $v \in X$ and has at least $k-1$ neighbors in $S$ or $v \in V \setminus X$ has at least $k$ neighbors in $X$. The $k$-tuple domination number $\gamma_{\times k}(G)$ is the minimum cardinality of a $k$-tuple dominating set of $G$. Clearly, $\gamma(G) = \gamma_{\times 1}(G) \leq \gamma_{\times k}(G)$, while $\gamma_t(G) \leq \gamma_{\times 2}(G)$ and $\gamma_{\times k}(G)$ can be determined using the following optimization problem:

$$\begin{aligned} \gamma_{\times k}(G) = \min \sum_{v \in V} x_v & \\ \text{subject to} \quad \sum_{u \in N[v]} x_u \geq k, & \text{ for all } v \in V. \end{aligned}$$

Specifically, a $k$-tuple dominating set where $k = 2$ is called a double dominating set and in this work we determine double domination number $\gamma_{\times 2}(\Pi_n)$ of small Pell graphs.

A function $f \colon V \to \{-1, 1\}$ is called a signed dominating function if $\sum_{u \in N[v]} f(u) \geq 1$ holds for every $v \in V$ [4]. The signed domination number $\gamma_s(G)$ of $G$ is the minimum of $\sum_{v \in V} f(v)$ taken over all signed dominating functions $f$ of $G$ and it can be determined using the following optimization problem [2]:

$$\begin{aligned} \gamma_s(G) = \min \sum_{v \in V} (2x_v - 1) & \\ \text{subject to} \quad \sum_{u \in N[v]} (2x_u - 1) \geq 1, & \text{ for all } v \in V. \end{aligned}$$

Here we note that binary variables $x_v$ associated with every vertex $v \in V$ indicates whether $v$ is assigned weight 1 ($x_v = 1$) or $-1$ ($x_v = 0$).

The connected domination number $\gamma_c(G)$ is the order of a smallest dominating set that induces a connected graph. We used the Miller-Tucker-Zemlin constraints to find a minimal connected domination set for Pell graphs [7].

The paired domination number $\gamma_p(G)$ is the order of a smallest dominating set $S \subseteq V$ s.t. the graph induced by $S$ contains a perfect matching. We associate to each edge $e = uv \in E$ a binary variable $x_e = x_{uv}$ indicating whether $e$ is present in the graph induced by a paired dominating set. Then the following optimization problem determines $\gamma_p(G)$ [2]:

$$\begin{aligned} \gamma_p(G) = 2 \cdot \min \sum_{e \in E} x_e & \\ \text{subject to} \quad & \sum_{u \in N(v)} x_{uv} \leq 1, \text{ for all } v \in V \\ & \sum_{u \in N(v)} \sum_{w \in N(u)} x_{uw} \geq 1, \text{ for all } v \in V. \end{aligned}$$

Using the integer linear programming approaches described in this section, we obtain the values and estimates of $\rho(\Pi_n)$, $i(\Pi_n)$, $\gamma_{\times 2}(\Pi_n)$, $\gamma_s(\Pi_n)$, $\gamma_c(\Pi_n)$, $\gamma_p(\Pi_n)$ for some small values of $n$ and collect these results in Table 4. Furthermore, in Tables 5, 6, 7, 9 and 10 in Appendix, we present example of a set of vertices giving $\rho(\Pi_n)$ and $\gamma_p(\Pi_n)$ for $n = 7$, $\gamma_c(\Pi_n)$ for $n = 5$, and $i(\Pi_n)$ and $\gamma_{\times 2}(\Pi_n)$ for $n = 6$ that were obtained during the computation of these values. In Table 8, we also present the set of vertices $v \in V(\Pi_6)$ for which $f(v) = -1$, where $f$ is a signed dominating function giving $\gamma_s(\Pi_6) = 45$.

Table 4: Values of additional domination type parameters for small Pell graphs.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\lvert V(\Pi_n)\rvert$ | 2 | 5 | 12 | 29 | 70 | 169 | 408 |
| $\rho(\Pi_n)$ | 1 | 2 | 3 | 6 | 11 | 22 | 46 |
| $i(\Pi_n)$ | 1 | 2 | 4 | 7 | 15 | 31 | 60–69 |
| $\gamma_{\times 2}(\Pi_n)$ | 2 | 4 | 7 | 13 | 27 | 56 | 113–121 |
| $\gamma_s(\Pi_n)$ | 2 | 3 | 4 | 11 | 20 | 45 | 88–102 |
| $\gamma_c(\Pi_n)$ | 1 | 2 | 4 | 9 | 18 | 35–38 | 66–82 |
| $\gamma_p(\Pi_n)$ | 2 | 2 | 4 | 10 | 16 | 34 | 72 |

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## Appendix

Table 5: Example of a 2-packing set for $\Pi_7$.

| |
|---|
| 0000110, 0001001, 0010000, 0010122, 0011221, 0012201, 0022022, 0022110, 0100022, 0100221, 0101100, 0102222, 0110101, 0111011, 0122000, 0220010, 0221122, 0221220, 0222200, 1000101, 1001022, 1002210, 1010011, 1010220, 1011100, 1012222, 1022001, 1100010, 1101111, 1122122, 1122220, 1220022, 1220100, 1220221, 1221001, 1222211, 2200122, 2200220, 2201000, 2202201, 2210001, 2211022, 2211221, 2212210, 2222010, 2222101. |

Table 6: Example of a minimal independent dominating set for $\Pi_6$.

| |
|---|
| 000000, 000221, 001011, 001122, 001220, 002200, 010011, 010110, 010122, 011101, 012211, 022000, 022022, 022220, 100022, 100101, 100110, 101000, 102211, 111001, 111010, 111100, 111221, 112222, 122111, 220000, 220111, 220220, 221022, 222201, 222210. |

Table 7: Example of a minimal double dominating set for $\Pi_6$.

| |
|---|
| 000000, 000022, 000111, 000220, 001022, 001100, 001101, 001110, 002211, 010000, 010010, 010101, 010220, 011011, 011101, 011122, 011221, 012210, 012211, 022000, 022011, 022022, 022110, 022220, 100001, 100110, 100111, 101001, 101010, 101221, 102200, 102211, 102222, 110022, 110100, 110122, 111010, 111022, 111122, 111220, 111221, 112200, 122000, 122101, 122111, 220001, 220011, 220100, 220220, 220221, 221001, 221010, 221110, 221122, 222201, 222211. |

Table 8: The set of vertices $v \in V(\Pi_6)$ for which $f(v) = -1$, where $f$ is a signed dominating function giving $\gamma_s(\Pi_6) = 45$.

| |
|---|
| 000001, 000010, 000101, 000110, 000122, 001010, 001022, 001101, 001110, 001221, 002222, 010001, 010100, 010122, 010220, 011001, 011011, 011100, 011111, 011221, 012200, 012211, 022010, 022022, 022100, 022122, 022220, 100000, 100011, 100111, 100221, 101000, 101022, 101101, 101110, 102201, 102210, 102222, 110000, 110011, 110100, 110111, 110220, 111011, 111110, 111111, 122001, 122010, 122101, 122220, 220010, 220022, 220101, 220122, 220220, 221001, 221010, 221101, 221221, 222200, 222210, 222222. |

Table 9: Example of a minimal connected dominating set for $\Pi_5$.

| |
|---|
| 00011, 00101, 00111, 00221, 01000, 01111, 01122, 02201, 02211, 10011, 11000, 11100, 11110, 11111, 11122, 11220, 22011, 22111. |

Table 10: Example of a minimal paired dominating set for $\Pi_7$.

| |
|---|
| 0000000, 0000100, 0000220, 0001022, 0001122, 0001220, 0010111, 0011001, 0011010, 0011100, 0011111, 0012200, 0022001, 0022220, 0100011, 0100111, 0101101, 0102201, 0110022, 0111110, 0112222, 0122011, 0122111, 0220000, 0220111, 0221000, 0221110, 0221111, 1000011, 1000111, 1001101, 1002210, 1002211, 1010010, 1011010, 1011101, 1022022, 1022122, 1022220, 1101000, 1101010, 1101221, 1102210, 1110001, 1110010, 1110022, 1110100, 1110101, 1110122, 1110220, 1110221, 1111221, 1112222, 1122000, 1122100, 1220220, 1221011, 1221022, 1222200, 1222201, 2200110, 2200111, 2201000, 2201022, 2201122, 2202201, 2210001, 2211110, 2211220, 2212201, 2222011, 2222111. |

Table 11: Example of a dominating set having $64$ vertices for $\Pi_7$.

| |
|---|
| 0000010, 0001001, 0002211, 0010022, 0010101, 0010221, 0011100, 0011110, 0022010, 0022122, 0100000, 0100111, 0101022, 0101220, 0102200, 0110000, 0111100, 0111110, 0112222, 0122001, 0122221, 0220000, 0220122, 0220220, 0221011, 0222201, 1000001, 1000100, 1000122, 1000220, 1001122, 1001221, 1002210, 1011000, 1011011, 1012201, 1022101, 1022220, 1101010, 1101101, 1110010, 1110011, 1110110, 1110111, 1112222, 1122000, 1122022, 1220101, 1221000, 1221022, 1221221, 1222210, 2200022, 2200100, 2200221, 2201010, 2202211, 2210001, 2211001, 2211122, 2211220, 2212200, 2222110, 2222111. |